

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

An Introduction to Applied Maths

Date: Monday, 17 May 2021

Time: 09:00 to 11:00

Time Allowed: 2 hours

Upload Time Allowed: 30 minutes

This paper has 4 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

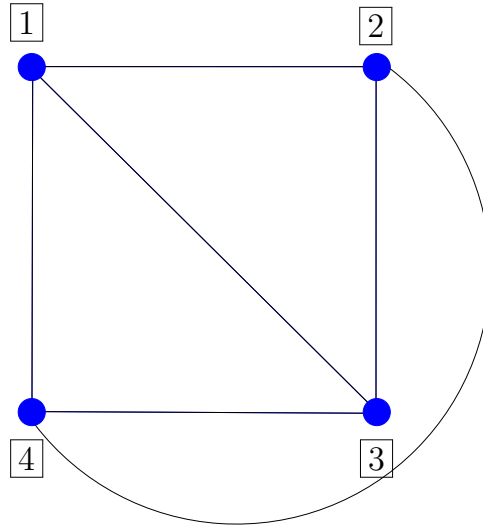
Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Consider the graph shown in the figure where large blue dots indicate nodes and all lines are edges:



- (a) Write down the Laplacian matrix $\mathbf{K}$ for this graph using the numbering of nodes shown to number the columns. (3 marks)
- (b) Verify that

$$\mathbf{e}_j = \begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ \omega^{3j} \end{pmatrix}, \quad j = 0, 1, 2, 3, \quad \omega = e^{2\pi i/4},$$

are (complex-valued) eigenvectors of $\mathbf{K}$ and find the eigenvalue associated with each eigenvector. (5 marks)

- (c) Now think of this graph as an electric circuit with each edge between nodes having unit conductance. Node $\boxed{1}$ is now set to unit voltage, and node $\boxed{3}$ is grounded (set to zero voltage) while Kirchhoff's current law holds at all other nodes. What is the effective conductance of this circuit? (5 marks)
- (d) Suppose the vector of voltages at the nodes for the problem in part (c) is written as

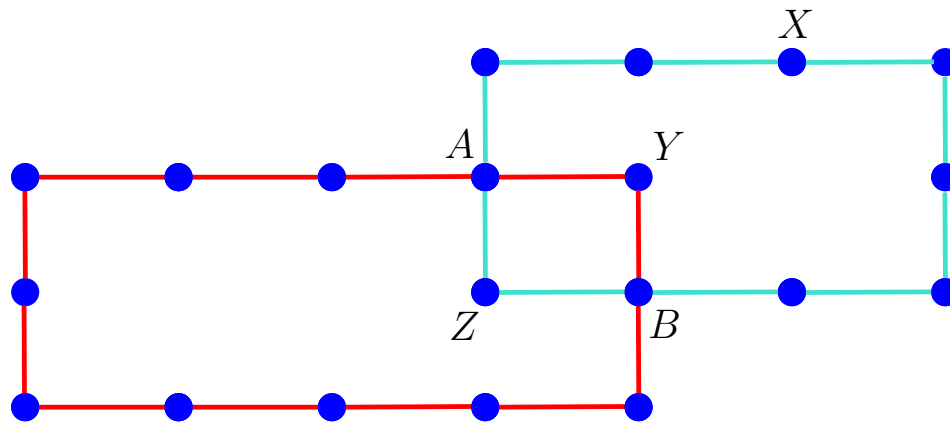
$$\mathbf{x} = \begin{pmatrix} 1 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} = \sum_{j=0}^3 a_j \mathbf{e}_j.$$

Find the coefficients a_0, a_1, a_2 and a_3 .

(7 marks)

(Total: 20 marks)

2. A tourist takes a random journey on the subway in a city where, at each station, the tourist gets off the train and decides at random which train line to take next, with each line having equal probability of being chosen. A map of the subway is as follows:

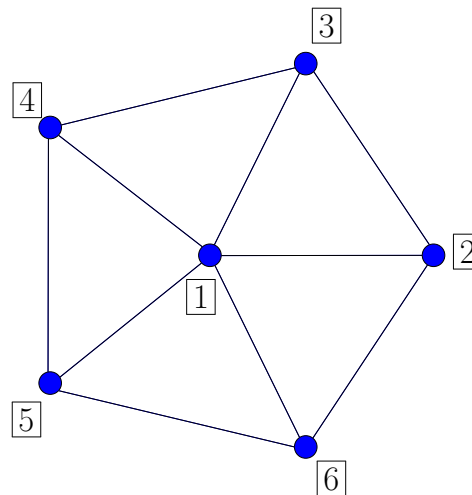


The (blue) dots are the subway stations. Lines between them are train lines.

- Find the probability of the tourist reaching station A before station B if he starts his journey at station X. (5 marks)
- If the tourist starts the journey at station A and leaves, what is the probability he will reach station B before returning to station A? (8 marks)
- If the tourist again starts the journey at station A and leaves, what is the probability that he reaches any of the stations B, Y or Z before returning to station A? (4 marks)
- The probability you calculated in part (c) should be greater than $1/2$. How could you deduce this before starting the calculation? (3 marks)

(Total: 20 marks)

3. This question concerns the following graph:



- Find the dimension of the left null space of the incidence matrix associated with this graph and explain your answer. (2 marks)
- Let $\mathbf{K}$ denote the Laplacian matrix of the graph. Find the dimension of the left null space of $\mathbf{K}$ and explain your answer. (2 marks)
- Write down the matrix $\mathbf{K}$ explicitly. (Column j of the matrix $\mathbf{K}$ **must** correspond to node $\boxed{j}$ shown in the figure). (3 marks)
- Show that 4 of the 6 real eigenvectors of $\mathbf{K}$ have the form

$$\mathbf{e}_j = \begin{pmatrix} 0 \\ \mathbf{x}_j \end{pmatrix}, \quad j = 1, 2, 3, 4,$$

where $\mathbf{x}_j$ is a 5-dimensional vector and find $\{\mathbf{e}_j | j = 1, 2, 3, 4\}$ explicitly. (10 marks)

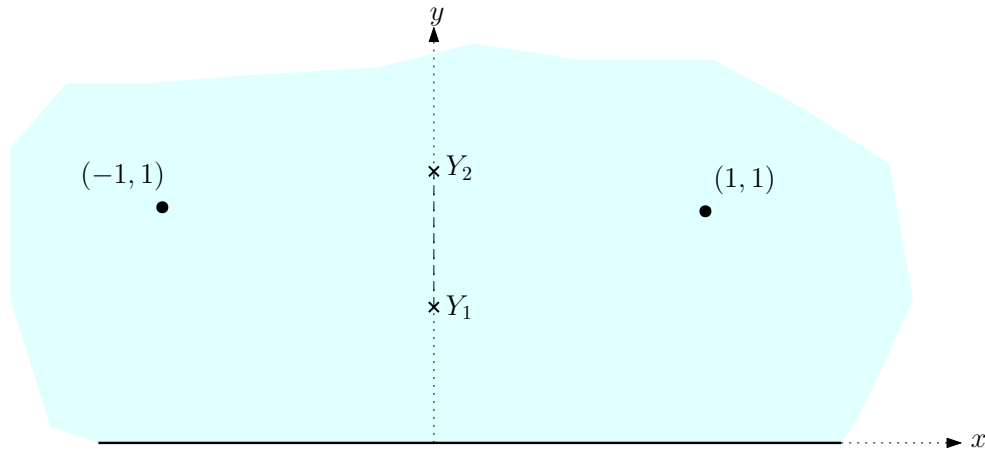
- Use the fact that the 2 remaining eigenvectors of $\mathbf{K}$, denoted by $\mathbf{e}_j$ for $j = 5, 6$, have the form

$$\mathbf{e}_j = \begin{pmatrix} a \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad j = 5, 6$$

for some real parameter a to find these 2 eigenvectors $\{\mathbf{e}_j | j = 5, 6\}$ explicitly. (3 marks)

(Total: 20 marks)

4. Consider an infinite conductor, of uniform conductivity $\hat{c} = 1$, occupying the upper half plane $y > 0$ in an (x, y) plane as shown in the Figure.



A current flows in the conductor with associated voltage potential $\phi(x, y)$ given by

$$\phi(x, y) = \text{Re}[h(z)], \quad z = x + iy,$$

where $\text{Re}[\cdot]$ means take a real part of a complex quantity and

$$h(z) = -\frac{m}{2\pi} \log \left[\frac{(z - a)(z - \bar{a})}{(z + a)(z + \bar{a})} \right], \quad a = \sqrt{2}e^{i\pi/4},$$

where m is a positive real constant.

- Show that there is a point source of current of strength m in the conductor at $(1, 1)$, a point source of current of strength $-m$ in the conductor at $(-1, 1)$ and no other singularities in the conductor. (4 marks)
- Show that no current flows across the boundary $y = 0$ into the lower half plane $y < 0$. (5 marks)
- Show that the total current flowing across the interval $[Y_1, Y_2]$ on the y axis (shown as a dashed line in the Figure) from the first quadrant to the second quadrant can be written as

$$\frac{m}{\pi} \int_{Y_1}^{Y_2} \left[\frac{1}{y^2 - 2y + 2} + \frac{1}{y^2 + 2y + 2} \right] dy.$$

(5 marks)

- Use the result of part (c) to calculate the total current flowing across the line $x = 0, y > 0$ from the first quadrant of the (x, y) plane to the second quadrant. (6 marks)

(Total: 20 marks)

1. Consider the graph shown in the figure where large blue dots indicate nodes and all lines with arrows are edges:

(a) The Laplacian is

$$\mathbf{K} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

meth seen ↓

(b) It is circulant, therefore it is known to have the eigenvectors given. By direct matrix multiplication,

3, A

$$\mathbf{K}\mathbf{e}_j = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ \omega^{3j} \end{pmatrix} = \lambda_j \begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ \omega^{3j} \end{pmatrix},$$

where

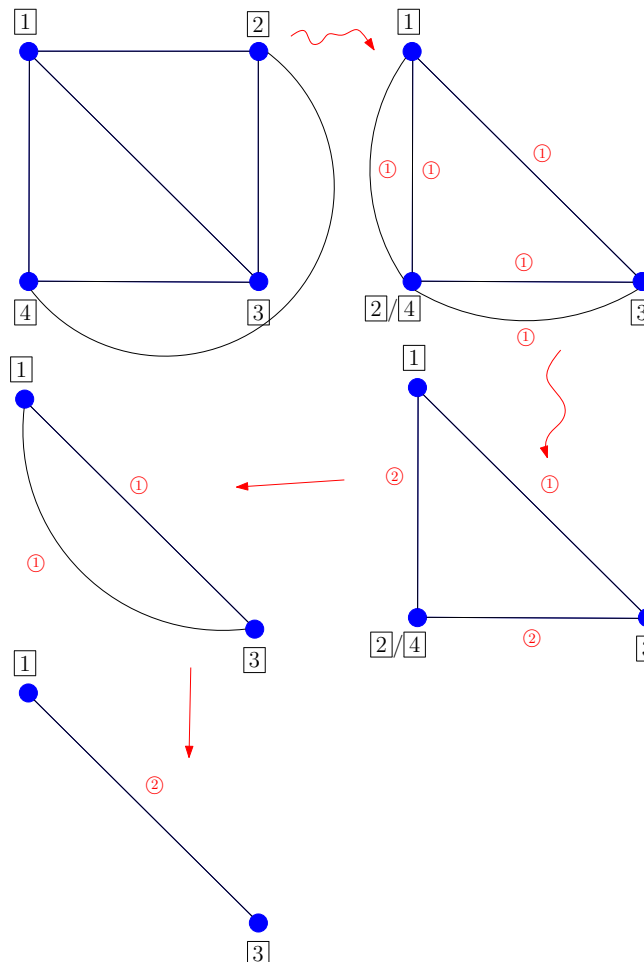
$$\lambda_j = 3 - \omega^j - \omega^{2j} - \omega^{3j}, \quad j = 0, 1, 2, 3. \quad (1)$$

meth seen ↓

This gives values 0, 4, 4, 4.

(c) From the following sequence of equivalent circuits:

5, A



we find

$$C_{\text{eff}} = 2.$$

Here we have used the “symmetry” of the circuit to deduce that the voltages at nodes 2 and 4 must be the same and, hence, they can be merged and the edge between them eliminated. The rules for conductors in series and parallel can then be used as indicated in the figure.

meth seen ↓

(d) The linear system associated with this circuit can be written

5, B

$$\mathbf{K}\mathbf{x} = \mathbf{f} = C_{\text{eff}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \quad (2)$$

Thus

$$\mathbf{K} \sum_{j=0}^3 a_j \mathbf{e}_j = \sum_{j=0}^3 a_j \lambda_j \mathbf{e}_j = C_{\text{eff}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \quad (3)$$

It is easy to check that the system $\{\mathbf{e}_j | j = 0, 1, 2, 3\}$ is orthonormal, i.e.,

$$\overline{\mathbf{e}_k}^T \mathbf{e}_j = 4\delta_{jk} \quad (4)$$

Using this, we can multiply (3) by $\overline{\mathbf{e}_k}^T$ to find that a_0 is not determined but

$$4a_k \lambda_k = C_{\text{eff}}(1 - \omega^{2k}), \quad k = 1, 2, 3, \quad (5)$$

or

$$a_k = \frac{C_{\text{eff}}(1 - \omega^{2k})}{4\lambda_k}, \quad k = 1, 2, 3. \quad (6)$$

Hence, from (6),

$$a_k = \frac{(1 - \omega^{2k})}{8}, \quad k = 1, 2, 3, \quad (7)$$

or

$$a_1 = \frac{1}{4}, \quad a_2 = 0, \quad a_3 = \frac{1}{4}.$$

Since $x_1 = 1$ and $x_3 = 0$, we also have

$$a_0 + a_1 + a_2 + a_3 = 1, \quad a_0 + \omega^2 a_1 + \omega^4 a_2 + \omega^6 a_3 = 0. \quad (8)$$

The first equation of these implies

$$a_0 = 1 - a_1 - a_2 - a_3 = \frac{1}{2}.$$

We did not need to use the second of the equations (8) because we used the value of C_{eff} found using “equivalent circuits”.

meth seen ↓

7, C

Additional note: If we had **not** used the “equivalent circuit” approach to find C_{eff} we would have to use the second equation in (8). Subtracting the two equations (8) gives

$$a_1(1 - \omega^2) + a_2(1 - \omega^4) + a_3(1 - \omega^6) = 1. \quad (9)$$

On substituting (6) into this equation we find

$$\frac{C_{\text{eff}}(1 - \omega^2)}{16}(1 - \omega^2) + \frac{C_{\text{eff}}(1 - \omega^4)}{16}(1 - \omega^4) + \frac{C_{\text{eff}}(1 - \omega^6)}{16}(1 - \omega^6) = 1 \quad (10)$$

This gives

$$C_{\text{eff}} [(1 - \omega^2)^2 + (1 - \omega^4)^2 + (1 - \omega^6)^2] = 16 \quad (11)$$

or

$$C_{\text{eff}} = 2, \quad (12)$$

thereby confirming the derivation through equivalent circuits. Hence, from (6),

$$a_k = \frac{(1 - \omega^{2k})}{8}, \quad k = 1, 2, 3, \quad (13)$$

or

$$a_1 = \frac{1}{4}, \quad a_2 = 0, \quad a_3 = \frac{1}{4}.$$

Finally, the first equation in (8) implies

$$a_0 = 1 - a_1 - a_2 - a_3 = \frac{1}{2}.$$

2. (a). The line of edges of the graph between stations A and B that passes through station X is a linear graph with 8 train lines. In an electric circuit analogy, if the voltage at station A is unity, and that at station B is zero, then the voltage will decrease by $1/8$ from station A as you progress to station B . Therefore the voltage at station X must be

$$1 - 3 \times \frac{1}{8} = \frac{5}{8}.$$

Within the analogy between electric circuits and random walks on graphs, this voltage is the probability of starting at station X and reaching A before B .

meth seen ↓

5, A

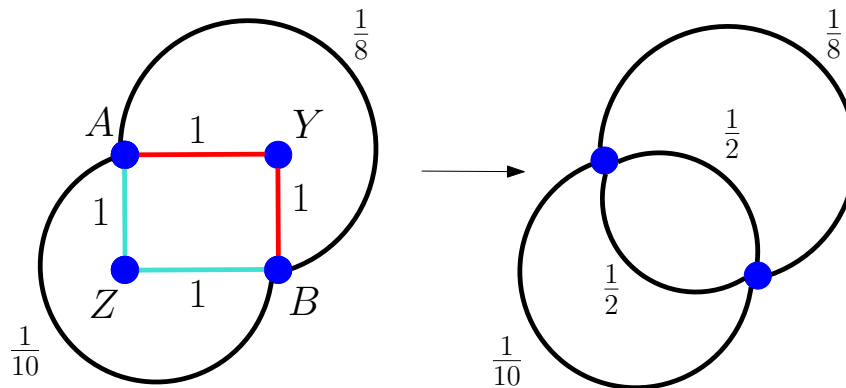
- (b) The effective conductance $C_{\text{eff}}^{(1)}$ of the loop of unit conductance conductors between stations A and B satisfies

$$\frac{1}{C_{\text{eff}}^{(1)}} = 8$$

where we have used the fact that *resistances* of conductors in series add up. Similarly, the effective conductance $C_{\text{eff}}^{(2)}$ of the *other* loop of 10 unit conductance conductors between stations A and B satisfies

$$\frac{1}{C_{\text{eff}}^{(2)}} = 10$$

The effective conductance between station A and B is therefore equivalent to this circuit:



where we have twice used the rule for two conductors in series:

$$\frac{c_a c_b}{c_a + c_b}$$

with $c_a = 1 = c_b$. The effective conductance C_{eff} is therefore the sum

$$C_{\text{eff}} = \frac{1}{10} + \frac{1}{2} + \frac{1}{2} + \frac{1}{8} = \frac{98}{80}.$$

The required escape probability p_{esc} is therefore given by

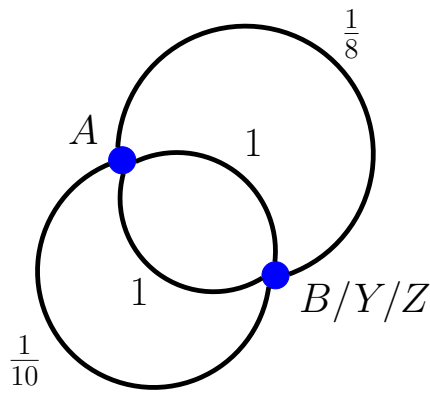
$$p_{\text{esc}} = \frac{C_{\text{eff}}}{4} = \frac{49}{160}.$$

since the number of edges attached to station A is 4.

meth seen ↓

8, B

- (c) We now put all the “escape stations”, B, Y and Z to zero potential, so they can all be merged in the electric circuit analogy (they are at the same voltage). Then the equivalent circuit is:



Using the fact the effective conductance of conductors in parallel add then the effective conductance $\hat{C}_{\text{eff}}$ is the sum

$$\hat{C}_{\text{eff}} = \frac{1}{10} + 1 + 1 + \frac{1}{8} = \frac{178}{80}.$$

The required escape probability $\hat{p}_{\text{esc}}$ is therefore given by

$$\hat{p}_{\text{esc}} = \frac{\hat{C}_{\text{eff}}}{4} = \frac{89}{160}$$

since the number of edges attached to station A is 4.

meth seen ↓

4, B

(d) Note that $89/160 > 80/160 = 1/2$ which could have been predicted because there are two edges from station A joining directly to the escape nodes Y and Z , which means that the probability of escaping in the first leg of the journey is $1/2$ (there are 4 routes, so probability $1/2$ of taking any 2 of them). But there are also *other* possible ways to reach stations B , Y and Z so the probability of escaping must be greater than $1/2$.

meth seen ↓

3, A

3. (a). There are $m = 10$ edges and $n = 6$ nodes so the incidence matrix, $\mathbf{A}$ say, is 10-by-6. The graph is connected, so the dimension of its right null space is 1. By the rank-nullity theorem, the rank r is therefore

$$r = n - 1 = 5.$$

The dimension of the left null space is therefore

$$m - r = 10 - 5 = 5.$$

Alternatively, the left null space of the incidence matrix is known to correspond to closed loops in the graph and, by inspection of the geometry of the graph, it is clear that there are 5 independent loops.

meth seen ↓

2, A

- (b). The Laplacian matrix $\mathbf{K} = \mathbf{A}^T \mathbf{A}$ is 6-by-6. Because the graph is connected the only right null vector is $(1, 1, 1, 1, 1, 1)^T$; its rank is 5. Since the matrix is square, by the rank-nullity theorem, the dimension of the left null space equals the dimension of the right null space, i.e. 1.

meth seen ↓

2, A

- (c). Using the usual rules of construction, and with column numbers corresponding to the node numbers in the figure, the Laplacian matrix is

$$\mathbf{K} = \begin{pmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & 0 & -1 & 3 \end{pmatrix}.$$

meth seen ↓

3, A

- (d). Note the sub-block form of $\mathbf{K}$:

$$\mathbf{K} = \begin{pmatrix} 5 & -\mathbf{x}_0^T \\ -\mathbf{x}_0 & \mathbf{C}_5 \end{pmatrix}, \quad \mathbf{x}_0 = (1, 1, 1, 1, 1)^T$$

and

$$\mathbf{C}_5 = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 3 \end{pmatrix}.$$

which, importantly, is a 5-by-5 circulant matrix. If

$$\hat{\mathbf{e}}_j = \begin{pmatrix} 0 \\ \mathbf{x}_j \end{pmatrix},$$

then

$$\mathbf{K} \hat{\mathbf{e}}_j = \begin{pmatrix} -\mathbf{x}_0^T \mathbf{x}_j \\ \mathbf{C}_5 \mathbf{x}_j \end{pmatrix}.$$

However, if we pick

$$\mathbf{x}_j = \begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ \omega^{3j} \\ \omega^{4j} \end{pmatrix}, \quad j = 1, 2, 3, 4, \quad \omega = e^{2\pi i/5},$$

then these are all known to be eigenvectors of $\mathbf{C}_5$, i.e.,

$$\mathbf{C}_5 \mathbf{x}_j = \lambda_j \mathbf{x}_j \quad (14)$$

with corresponding eigenvalues (found by direct substitution):

$$\lambda_j = 3 - 2 \cos(2\pi j/5), \quad j = 1, 2, 3, 4. \quad (15)$$

Moreover,

$$\mathbf{x}_0^T \mathbf{x}_j = \sum_{k=0}^4 [\omega^j]^k. \quad (16)$$

But using the sum of a finite geometric progression

$$\mathbf{x}_0^T \mathbf{x}_j = \frac{1 - (\omega^j)^5}{1 - \omega^j} = 0, \quad j = 1, 2, 3, 4. \quad (17)$$

Note that the last relation fails if $j = 0$, but this case has been excluded. Therefore

$$\mathbf{K} \hat{\mathbf{e}}_j = \begin{pmatrix} 0 \\ \lambda_j \mathbf{x}_j \end{pmatrix} = \lambda_j \hat{\mathbf{e}}_j, \quad j = 1, 2, 3, 4$$

which shows that $\{\hat{\mathbf{e}}_j | j = 1, 2, 3, 4\}$ are complex-valued eigenvectors. Since $\mathbf{K}$ is real (and symmetric) their real and imaginary parts must be real eigenvectors. The real and imaginary parts of $\hat{\mathbf{e}}_j$ for $j = 1, 2$ are all that are needed because the real and imaginary parts of $\hat{\mathbf{e}}_j$ for $j = 3, 4$ give the same vectors. Therefore, the 4 vectors

$$\begin{pmatrix} 0 \\ 1 \\ \cos(2\pi j/5) \\ \cos(4\pi j/5) \\ \cos(6\pi j/5) \\ \cos(8\pi j/5) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sin(2\pi j/5) \\ \sin(4\pi j/5) \\ \sin(6\pi j/5) \\ \sin(8\pi j/5) \end{pmatrix}, \quad j = 1, 2$$

are the required 4 real eigenvectors $\{\mathbf{e}_k | k = 1, 2, 3, 4\}$. The corresponding eigenvalues are

$$\lambda_j = 3 - 2 \cos(2\pi j/5), \quad j = 1, 2$$

and each of these has multiplicity 2 (giving 4 of the eigenvalues).

(e). If we are *given* that other eigenvectors have the form

$$\mathbf{e}_j = \begin{pmatrix} a \\ \mathbf{x}_0 \end{pmatrix}, \quad \mathbf{x}_0 = (1, 1, 1, 1, 1)^T$$

unseen ↓

10, D

then, using the sub-block decomposition of $\mathbf{K}$

$$\mathbf{K}\mathbf{e}_j = \mathbf{K} \begin{pmatrix} a \\ \mathbf{x}_0 \end{pmatrix} = \begin{pmatrix} 5a - \mathbf{x}_0^T \mathbf{x}_0 \\ -a\mathbf{x}_0 + \mathbf{C}_5 \mathbf{x}_0 \end{pmatrix} = \begin{pmatrix} 5a - 5 \\ (1-a)\mathbf{x}_0 \end{pmatrix} = \begin{pmatrix} \frac{5a-5}{a} \times a \\ (1-a) \times \mathbf{x}_0 \end{pmatrix}.$$

This means that, if these are to be eigenvectors with eigenvalue λ , we must have

$$\frac{5a-5}{a} = (1-a) = \lambda.$$

The first equation gives a quadratic equation for a :

$$a^2 + 4a - 5 = 0$$

which has roots

$$a = 1, -5.$$

The two remaining eigenvectors are

$$\mathbf{e}_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{e}_6 = \begin{pmatrix} -5 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The corresponding eigenvalues are, respectively,

$$\lambda_5 = 0, \quad \lambda_6 = 6.$$

unseen ↓

3, B

4. (a). The complex potential can be written

$$h(z) = \underbrace{-\frac{m}{2\pi} \log(z-a)}_{\text{current source } m \text{ at } a} + \underbrace{\frac{m}{2\pi} \log(z+\bar{a})}_{\text{current source } -m \text{ at } -\bar{a}} - \underbrace{\frac{m}{2\pi} \log \left[\frac{z-\bar{a}}{z+a} \right]}_{\text{analytic in } y>0}$$

so it has the correct functional form for a current source of strength m at $z = a = \sqrt{2} \times (1+i)/\sqrt{2} = 1+i$ or $(1, 1)$ and a current source of strength $-m$ at $z = -\bar{a} = -1+i$ or $(-1, 1)$. The last term in this expression is analytic in the conductor $y > 0$ since its only singularities are at $\bar{a}$ and $-a$ which are in $y < 0$.

meth seen ↓

4, A

(b). We employ the formula

$$J^{(x)} - iJ^{(y)} = -\hat{c} \frac{dh}{dz}$$

well-known from lectures or, since $\hat{c} = 1$,

$$J^{(x)} - iJ^{(y)} = -\frac{dh}{dz} = \frac{m}{2\pi} \left[\frac{1}{z-a} + \frac{1}{z-\bar{a}} - \frac{1}{z+a} - \frac{1}{z+\bar{a}} \right].$$

If $z = x \in \mathbb{R}$ it is clear that the right hand side is real since the first two terms on the right hand side are complex conjugates of each other, as are the third and fourth terms, i.e.,

$$J^{(x)} - iJ^{(y)} = \frac{m}{2\pi} \left[\underbrace{\frac{1}{x-a} + \frac{1}{x-\bar{a}}}_{\text{sum of complex conjugates}} - \left(\underbrace{\frac{1}{x+a} + \frac{1}{x+\bar{a}}}_{\text{sum of complex conjugates}} \right) \right].$$

Therefore, on the real axis,

$$J^{(y)} = 0$$

which means there is no current flowing across this axis.

meth seen ↓

5, A

(c). On the other hand if $z = iy$, the formula above gives

$$\begin{aligned} J^{(x)} - iJ^{(y)} &= \frac{m}{2\pi} \left[\frac{1}{iy-a} + \frac{1}{iy-\bar{a}} - \frac{1}{iy+a} - \frac{1}{iy+\bar{a}} \right] \\ &= \frac{m}{2\pi} \left[\frac{1}{iy-a} - \frac{1}{iy+\bar{a}} + \frac{1}{iy-\bar{a}} - \frac{1}{iy+a} \right]. \end{aligned}$$

On combining the first two terms, and the second two terms, on the right hand side this can be written as

$$J^{(x)} - iJ^{(y)} = \frac{m}{2\pi} \left[\frac{a+\bar{a}}{-y^2 - iy(a-\bar{a}) - |a|^2} + \frac{a+\bar{a}}{-y^2 + iy(a-\bar{a}) - |a|^2} \right].$$

But

$$a + \bar{a} = 2, \quad a - \bar{a} = 2i, \quad |a|^2 = 2,$$

hence

$$J^{(x)} - iJ^{(y)} = -\frac{m}{\pi} \left[\frac{1}{y^2 - 2y + 2} + \frac{1}{y^2 + 2y + 2} \right].$$

Therefore the flux from right to left, in the $-x$ direction from the first quadrant to the second quadrant, is

$$\int_{Y_1}^{Y_2} -J^{(x)} dy = \frac{m}{\pi} \int_{Y_1}^{Y_2} \left[\frac{1}{y^2 - 2y + 2} + \frac{1}{y^2 + 2y + 2} \right] dy$$

which is the required answer.

meth seen ↓

5, C

(d). On completing the square in both denominators, and taking $Y_1 = 0, Y_2 = \infty$, the required total current is

$$\frac{m}{\pi} \int_0^\infty \left[\frac{1}{(y-1)^2 + 1} + \frac{1}{(y+1)^2 + 1} \right] dy.$$

We can do both integrals at once by considering

$$I(a) = \int_0^\infty \frac{dy}{(y-a)^2 + 1}$$

then the required current can be written as

$$\frac{m}{\pi} [I(1) + I(-1)].$$

To compute $I(a)$ make the substitution

$$\eta = y - a$$

then

$$I(a) = \int_{-a}^\infty \frac{d\eta}{(1+\eta^2)} = [\tan^{-1} \eta]_{-a}^\infty = \left[\frac{\pi}{2} - \tan^{-1}(-a) \right].$$

Therefore the required integral is

$$\frac{m}{\pi} \left[\left[\frac{\pi}{2} - \tan^{-1}(-1) \right] + \left[\frac{\pi}{2} - \tan^{-1}(1) \right] \right] = m,$$

where we have used the fact that $\tan^{-1}(-x) = -\tan^{-1}(x)$.

meth seen ↓

6, D

Additional note: this result can also be argued based on the fact that, by the conservation of current, and given that no current leaves the circuit through the x -axis (see part (b)), then all current entering the conductor at the source in the first quadrant must leave via the sink in the second quadrant, meaning that the current across the imaginary axis must be m . **However**, the question asks the student explicitly to use part (c) therefore only partial credit is given for the correct answer without the above calculation. (This alternative deduction can be used by students as a check on their calculation).

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Math40007	1	This question was answered well overall. There are multiple ways to approach the problem. In part (b) it was sufficient to point out the "circulant" property and quote the result established in lectures on the eigenvectors of circulant matrices. Many students did this; others checked directly. Part (c) can be approach in many different ways, either using "equivalent circuits", or by linear algebra; either way was acceptable provided full explanations were given. Similarly, solution to part (d) depended on how part (c) was solved: many students chose to use an orthogonality property to find the potentials, others found the potentials and fitted the coefficients a posterior.
Math40007	2	On the whole, this was a pretty easy and easy to understand question. I have seen very few scripts addressing the wrong question, such as considering again X as the starting point. There were plenty of scripts with all marks. Many students spent a lot of work and time on part a, which was actually pretty simple, as the potential drops linearly. Part b) required some algebra, which got some students confused. Part c) was similar to part b) and mostly done just as well. Part d) was very widely correctly answered. Some students would have benefitted from providing more details about their derivation.
Math40007	3	Parts a, b & c were pretty straightforward. Majority of marks were lost in part d, as most candidates did not develop the general solution given in lecture notes for a circulant matrix in as much detail as expected. Here quite a few ignored the imaginary part of the complex eigenvector, since matrix K is real and symmetric. Moreover multiplicity of solutions was also overlooked by quite a few. Some used software to find eigenvalues and eigenvectors for part d, but then could not develop the solution in the detail that was required, and so those that took this route scored on the low side.

Math40007	4	This question was done very well. Sadly a few students ran out of time.
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