

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

An Introduction to Applied Maths

Date: 23 May 2022

Time: 09:00 – 11:00 (BST)

Time Allowed: 2:00 hours

Upload Time Allowed: 30 minutes

This paper has 4 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

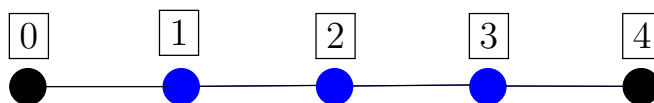
Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Consider the electric circuit involving 5 nodes and 4 conductors shown in this figure:



All edges are conductors with unit conductance. Throughout this question nodes $\boxed{0}$ and $\boxed{4}$, shown in black, are always grounded.

- (a) Suppose node $\boxed{1}$ is set to unit voltage with Kirchhoff's current law holding at nodes $\boxed{2}$ and $\boxed{3}$.

- (i) Find the vector of voltages

$$\mathbf{X}^{(1)} = \begin{pmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \\ \phi_3^{(1)} \end{pmatrix}$$

where $\phi_k^{(1)}$ denotes the voltage at node $\boxed{k}$. (2 marks)

- (ii) Find the effective conductance of this circuit (i.e. the current divergence at node $\boxed{1}$). (2 marks)

- (b) Suppose node $\boxed{2}$ is set to unit voltage with Kirchhoff's current law holding at nodes $\boxed{1}$ and $\boxed{3}$.

- (i) Find the vector of voltages

$$\mathbf{X}^{(2)} = \begin{pmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \\ \phi_3^{(2)} \end{pmatrix}$$

where $\phi_k^{(2)}$ denotes the voltage at node $\boxed{k}$. (2 marks)

- (ii) Find the effective conductance of this circuit (i.e. the current divergence at node $\boxed{2}$). (2 marks)

Question continues on the next page

- (c) Suppose node $\boxed{3}$ is set to unit voltage with Kirchhoff's current law holding at nodes $\boxed{1}$ and $\boxed{2}$.

(i) Find the vector of voltages

$$\mathbf{X}^{(3)} = \begin{pmatrix} \phi_1^{(3)} \\ \phi_2^{(3)} \\ \phi_3^{(3)} \end{pmatrix}$$

where $\phi_k^{(3)}$ denotes the voltage at node $\boxed{k}$. (2 marks)

- (ii) Find the effective conductance of this circuit (i.e. the current divergence at node $\boxed{3}$). (2 marks)

- (d) Let $\mathbf{x}$ denote any 3-dimensional vector of voltages at nodes $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$ and let

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

be the corresponding 3-dimensional vector of current divergences at these nodes. Let $\hat{\mathbf{x}}$ and $\hat{\mathbf{f}}$ be any other such pair. Show that

$$\mathbf{x}^T \hat{\mathbf{f}} = \hat{\mathbf{x}}^T \mathbf{f}.$$

(4 marks)

- (e) Use the results of any of parts (a)–(d) to find the 3-by-3 matrix $\mathbf{M}$ such that

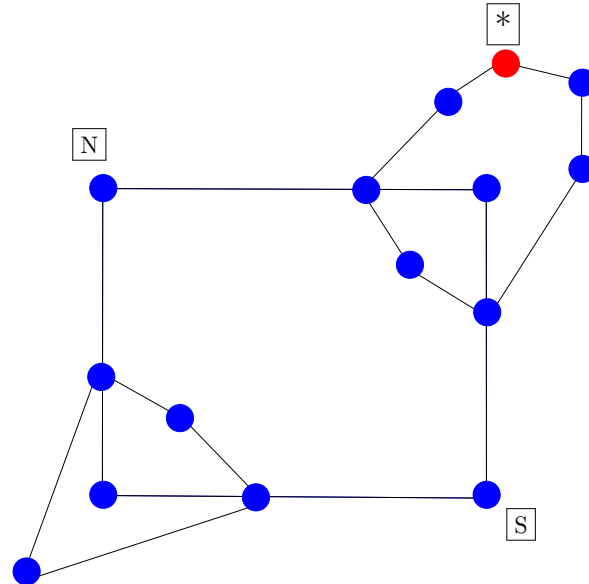
$$\mathbf{x} = \mathbf{M}\mathbf{f}$$

is the 3-dimensional vector of voltages at nodes $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$ corresponding to any given vector $\mathbf{f}$.

(4 marks)

(Total: 20 marks)

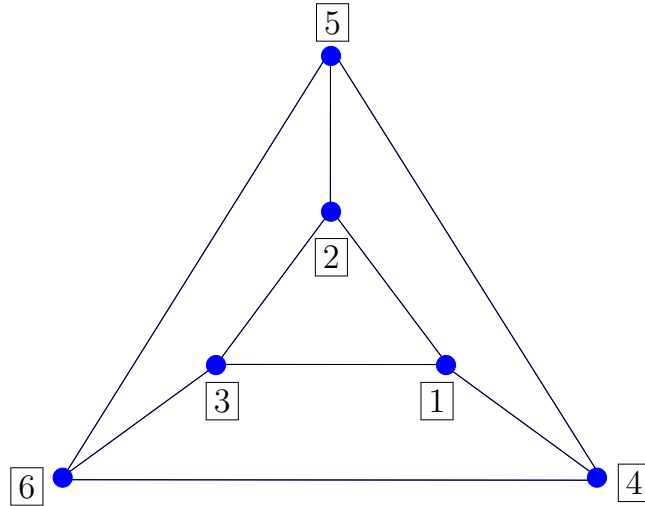
2. Consider the following subway map of a small city:



- (a) Let $\mathbf{A}$ be the incidence matrix of this subway system when viewed as a graph where the stations are nodes and the train lines between them are edges. Find the dimension of the left nullspace of $\mathbf{A}$? (4 marks)
- (b) A tourist leaves station $\boxed{\text{N}}$ on a random journey where, on arriving at each station, the tourist chooses at random which train line to take out of that station, with each train line at any station chosen with equal probability. What is the probability that, on such a journey, the tourist arrives at station $\boxed{\text{S}}$ before returning to station $\boxed{\text{N}}$? (8 marks)
- (c) Now suppose the tourist leaves on such a journey from the red node $\boxed{*}$. Find the probability that the tourist reaches station $\boxed{\text{N}}$ before station $\boxed{\text{S}}$. (8 marks)

(Total: 20 marks)

3. Consider the following graph:



- (a) Using the node numbering shown in the figure to number the corresponding rows and columns, write down the Laplacian matrix $\mathbf{K}$ of this graph. (2 marks)
- (b) Show that $\mathbf{K}$ can be written in the block matrix form

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{K}_3 \end{bmatrix},$$

where $\mathbf{I}_3$ is the 3-by-3 identity matrix and find the 3-by-3 matrix $\mathbf{K}_3$. (3 marks)

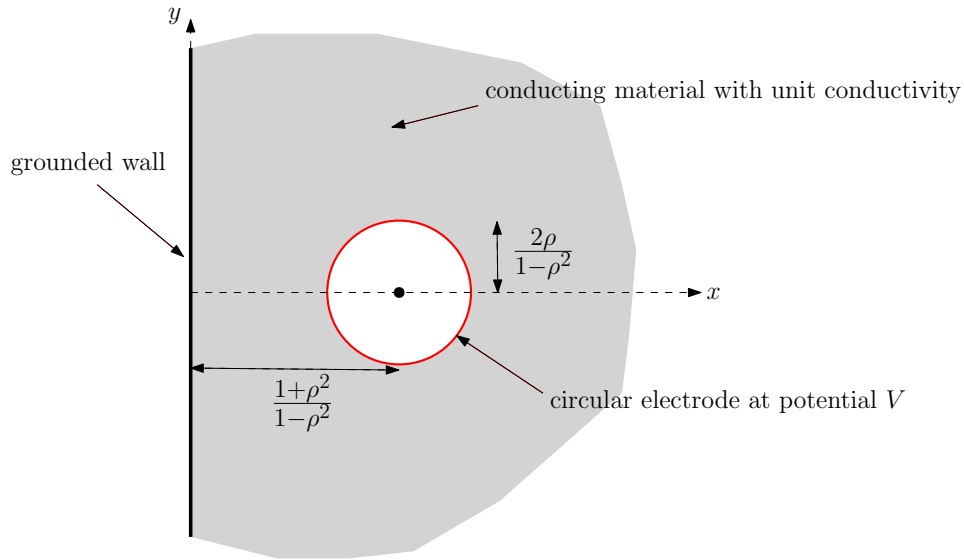
- (c) Find a set of linearly independent real-valued eigenvectors $\{\mathbf{e}_j | j = 1, 2, 3\}$ and the corresponding eigenvalues $\{\lambda_j | j = 1, 2, 3\}$ of $\mathbf{K}_3$. (5 marks)
- (d) By considering vectors of the form

$$\mathbf{x}_j = \begin{pmatrix} \mathbf{e}_j \\ a_j \mathbf{e}_j \end{pmatrix}, \quad j = 1, 2, 3$$

for a set of suitable real constants a_j , find the 6 eigenvalues of $\mathbf{K}$ and determine the corresponding eigenvectors. (10 marks)

(Total: 20 marks)

4. Consider the voltage distribution in a two-dimensional conductor of unit conductivity in an (x, y) -plane, shaded gray in the figure, generated by a circular electrode, centred at $\left(\frac{1+\rho^2}{1-\rho^2}, 0\right)$ and with radius $\frac{2\rho}{1-\rho^2}$ where $0 < \rho < 1$, and held at constant voltage V near a grounded half-plane with an infinite wall at $x = 0$ as shown here:



The voltage distribution $\phi(x, y)$ at any point (x, y) in the conductor is given by

$$\phi(x, y) = \operatorname{Re}[h(z)], \quad \text{where} \quad h(z) = \frac{V}{\log \rho} \log \left(\frac{1-z}{1+z} \right), \quad z = x + iy,$$

where $\operatorname{Re}[\cdot]$ means take a real part of the complex-valued quantity in square brackets.

- Verify that $\phi(x, y)$ satisfies the boundary condition that the wall at $x = 0$ is grounded. (3 marks)
- Verify that $\phi(x, y)$ satisfies the condition that the boundary of the circular electrode is held at constant voltage V . (6 marks)
- Find an expression for the current density distribution on the grounded wall along $x = 0$. (3 marks)
- At which point on the grounded wall $x = 0$ is the magnitude of the current density greatest? (2 marks)
- Calculate the total current leaving the conductor through the grounded wall along $x = 0$. (3 marks)
- Suppose that we simultaneously take the limits $\rho \rightarrow 0$ and $V \rightarrow \infty$ in such a way that the ratio $V/\log \rho$ remains fixed, equal to the constant

$$-\frac{m}{2\pi}.$$

Show that this electrode problem reduces to that of a point current source of strength m situated at $z = 1$ near a grounded wall along $x = 0$. (3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

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M40007: Introduction to Applied Mathematics
(Solutions)

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1. (a)(i) The easiest method is to use the harmonicity of the voltage potentials at the non-electrified nodes where Kirchhoff's current law (KCL) holds. For this part, KCL holds at nodes 2 and 3 yielding the equations

$$\phi_2^{(1)} = \frac{1 + \phi_3^{(1)}}{2}, \quad \phi_3^{(1)} = \frac{0 + \phi_2^{(1)}}{2}$$

which are easily solved to give

$$\phi_2^{(1)} = \frac{2}{3}, \quad \phi_3^{(1)} = \frac{1}{3}$$

Thus,

$$\mathbf{X}^{(1)} = \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

2, A

- (a)(ii) The effective conductance is the total current out of node 1 which, by Ohm's law, and using the fact that all conductances are unity, is

$$1 + \left(1 - \frac{2}{3}\right) = \frac{4}{3}$$

2, A

- (b)(i) Similar methods lead to

$$\phi_1^{(2)} = \frac{0 + 1}{2}, \quad \phi_3^{(2)} = \frac{0 + 1}{2}$$

so that

$$\mathbf{X}^{(2)} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

2, A

- (b)(ii) The effective conductance is the total current out of node 2 which, by Ohm's law, and using the fact that all conductances are unity, is

$$\left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) = \frac{1}{1}$$

2, A

- (c)(i) By symmetry, this result can be inferred immediately from part (a) since the circuit is the same with nodes 1 and 3 interchanged:

$$\mathbf{X}^{(3)} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}$$

2, A

(c)(ii) The effective conductance

$$1 + \left(1 - \frac{2}{3}\right) = \boxed{\frac{4}{3}}$$

2, A

(d) Let $\mathbf{A}$ be the incidence matrix of the 5-node graph. For a given vector $\mathbf{X}$ of voltage potentials the current divergence vector $\mathbf{F}$ is defined as

$$\mathbf{F} = -\mathbf{A}^T \mathbf{w}, \quad \mathbf{w} = -\mathbf{e}, \quad \mathbf{e} = \mathbf{A}\mathbf{X}$$

Similarly, for any other pair $\hat{\mathbf{X}}$ and $\hat{\mathbf{F}}$,

$$\hat{\mathbf{F}} = -\mathbf{A}^T \hat{\mathbf{w}}, \quad \hat{\mathbf{w}} = -\hat{\mathbf{e}}, \quad \hat{\mathbf{e}} = \mathbf{A}\hat{\mathbf{X}}$$

Therefore

$$\mathbf{X}^T \hat{\mathbf{F}} = \mathbf{X}^T \mathbf{A}^T \mathbf{A} \hat{\mathbf{X}} = (\mathbf{A}\mathbf{X})^T (\mathbf{A}\hat{\mathbf{X}}) = (\mathbf{A}\hat{\mathbf{X}})^T (\mathbf{A}\mathbf{X}) = \hat{\mathbf{X}}^T \mathbf{A}^T \mathbf{A} \mathbf{X} = \hat{\mathbf{X}}^T \mathbf{F}.$$

Now set

$$\mathbf{X} = \begin{pmatrix} 0 \\ \mathbf{x} \\ 0 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} f_0 \\ \mathbf{f} \\ f_4 \end{pmatrix}, \quad \hat{\mathbf{X}} = \begin{pmatrix} 0 \\ \hat{\mathbf{x}} \\ 0 \end{pmatrix}, \quad \hat{\mathbf{F}} = \begin{pmatrix} \hat{f}_0 \\ \hat{\mathbf{f}} \\ \hat{f}_4 \end{pmatrix}$$

then

$$\mathbf{X}^T \hat{\mathbf{F}} = \mathbf{x}^T \hat{\mathbf{f}} = \hat{\mathbf{X}}^T \mathbf{F} = \hat{\mathbf{x}}^T \mathbf{f},$$

which gives the required result:

$$\mathbf{x}^T \hat{\mathbf{f}} = \hat{\mathbf{x}}^T \mathbf{f}.$$

4, C

(e) Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

be the required vector of potentials. To derive the required matrix $\mathbf{M}$ such that $\mathbf{x} = \mathbf{M}\mathbf{f}$ for some given $\mathbf{f}$ where

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

notice that by rescaling the solution $\mathbf{X}^{(1)}$ in part (a) by the effective conductance $4/3$, we get

$$\hat{\mathbf{x}}^{(1)} = \frac{3}{4} \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \quad \hat{\mathbf{f}}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which, when used in the result of part (d) with the choices

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}^{(1)}, \quad \hat{\mathbf{f}} = \hat{\mathbf{f}}^{(1)}$$

gives

$$x_1 = \hat{\mathbf{x}}^{(1)T} \mathbf{f}.$$

Similarly, using the analogously rescaled result of part (b), namely,

$$\hat{\mathbf{x}}^{(2)} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}, \quad \hat{\mathbf{f}}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

in part (d) we get

$$x_2 = \hat{\mathbf{x}}^{(2)T} \mathbf{f}$$

and, from the analogously rescaled result of part (c),

$$\hat{\mathbf{x}}^{(3)} = \frac{3}{4} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}, \quad \hat{\mathbf{f}}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

we get

$$x_3 = \hat{\mathbf{x}}^{(3)T} \mathbf{f}.$$

In other words, rewriting this in matrix form, we have deduced that

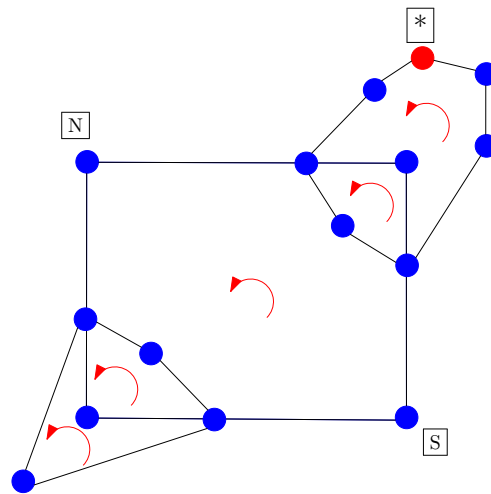
$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{M} \mathbf{f},$$

where

$$\mathbf{M} = \begin{bmatrix} \hat{\mathbf{x}}^{(1)T} \\ \hat{\mathbf{x}}^{(2)T} \\ \hat{\mathbf{x}}^{(3)T} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}.$$

4, B

2. The given subway map is



- (a) The matrix $\mathbf{A}$ is an m -by- n matrix where, counting nodes, $n = 15$, and, counting edges, $m = 19$. The graph is connected, so the dimensional of the right nullspace is 1 meaning, by the rank-nullity theorem, that the rank $r = n - 1 = 14$. The dimension of the left nullspace is therefore $m - r = 19 - 14 = 5$, also by the rank-nullity theorem.

Another possible route to the answer is to spot that there are 5 independent closed loops in the graph, as shown in the figure.

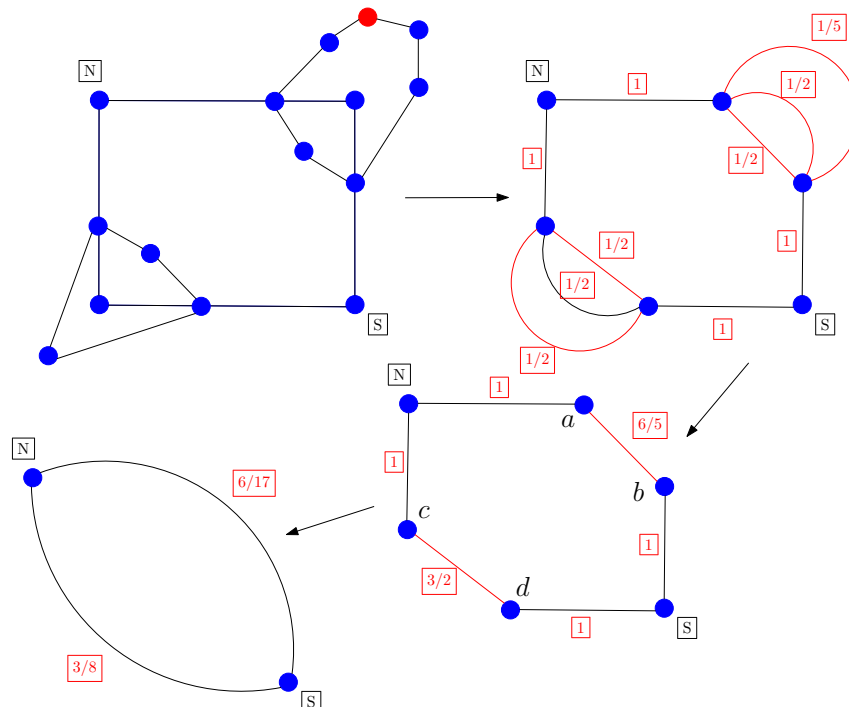
4, A

- (b) It is known from the electric circuit/random walk analogy explored in lectures that the probability required is equivalent to

$$\frac{C_{\text{eff}}}{2},$$

where C_{eff} is the effective conductance of a circuit with $\boxed{\text{N}}$ at unit voltage and $\boxed{\text{S}}$ grounded. The denominator 2 is the number of edges attached to station $\boxed{\text{N}}$.

The easiest way to compute C_{eff} is to reduce it to “equivalent circuits” as follows:



where we have used the two rules that the conductance of two conductors c_a and c_b in series is

$$\frac{c_a c_b}{c_a + c_b}$$

and that the conductance of two conductors c_a and c_b in parallel is

$$c_a + c_b$$

In the end, after the reductions shown in the figure,

$$C_{\text{eff}} = \frac{6}{17} + \frac{3}{8} = \frac{99}{136}.$$

Hence the required “escape probability” is

$$\frac{99}{272}.$$

8, C

- (c) Consider the values of the voltages a, b, c and d as indicated in the figure. By Ohm's law, and the results of part (a) which says that the current out of station N along the “upper route” is $6/17$, we have the equations

$$\frac{6}{17} = 1 - a, \quad b = \frac{6}{17}.$$

It follows easily that

$$a = \frac{11}{17}, \quad b = \frac{6}{17}.$$

Now the voltage drop of

$$a - b = \frac{5}{17}$$

between the nodes at potentials a and b must occur equally over the 5 conductors making up the loop in which the red node is the second along from the node at potential a . There is therefore a voltage drop of $1/17$ across each edge leading to the voltage at the red node being

$$\frac{11}{17} - 2 \times \frac{1}{17} = \frac{9}{17}.$$

By the electric circuit analogy, this is the probability starting at the red node of reaching N before S.

8, B

3. (a) By the usual rules for constructing a Laplacian we find

$$\mathbf{K} = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}.$$

Note: the sum of all elements of each row must be zero.

2, A

- (b) The Laplacian can be written in the block-matrix form

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{K}_3 \end{bmatrix},$$

where $\mathbf{I}_3$ is the 3-by-3 identity matrix and

$$\mathbf{K}_3 = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

3, A

- (c) There are at least 3 ways to do this part, all of them acceptable provided they are explained properly. One method is to notice that

$$\mathbf{K}_3 = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} = 4\mathbf{I}_3 - \mathbf{J},$$

where

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The matrix $\mathbf{J}$ is sufficiently simple that its eigenvectors and corresponding eigenvalues can be found by inspection to be

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \lambda_1 = 3, \quad \mathbf{e}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \lambda_2 = 0, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \lambda_3 = 0.$$

Consequently, since $\mathbf{K}_3 = 4\mathbf{I}_3 - \mathbf{J}$, the matrix $\mathbf{K}_3$ shares these eigenvectors, although the corresponding eigenvalues are different:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \lambda_1 = 1, \quad \mathbf{e}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \lambda_2 = 4, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \lambda_3 = 4.$$

5, B

Note: An alternative method, which is likely to be more popular, is to notice that $\mathbf{K}_3$ is a 3-by-3 **circulant matrix** so it is known from lectures that a set of its (complex) eigenvectors is

$$\mathbf{e}_j = \begin{pmatrix} 1 \\ \omega^{(j-1)} \\ \omega^{2(j-1)} \end{pmatrix}, \quad j = 1, 2, 3$$

with eigenvalues

$$\lambda_j = 3 - 2\cos(2\pi(j-1)/3), \quad j = 1, 2, 3$$

where

$$\omega = e^{2\pi i/3}.$$

These eigenvectors are

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \lambda_1 = 1, \quad \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \lambda_2 = 4, \quad \begin{pmatrix} 1 \\ \omega^2 \\ \omega^4 \end{pmatrix}, \lambda_3 = 4.$$

Now the question specifically asks for real-valued eigenvectors. So it is necessary to take real and imaginary parts of the above complex eigenvectors (and possibly rescale too) to get real eigenvectors. Using the facts that

$$\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \omega^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

we then find

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \lambda_1 = 1, \quad \mathbf{e}_2 = \begin{pmatrix} 1 \\ -1/2 \\ -1/2 \end{pmatrix}, \lambda_2 = 4, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \lambda_3 = 4.$$

(Important note for graders: note that these $\mathbf{e}_1$ and $\mathbf{e}_3$ are the same as those found using the previous method; the new choice of $\mathbf{e}_2$ is just a linear combination of the $\mathbf{e}_2$ and $\mathbf{e}_3$ found using the earlier method which is fine because these lie in the same eigenspace corresponding to $\lambda = 4$. Obviously, any other linear combination is also acceptable and it might be necessary to check any other $\lambda = 4$ eigenvectors given by a student that differ from those written above).

Note: a third possible method is to note that since $\mathbf{K}_3$ is just a 3-by-3 matrix its eigenvalues can alternatively be found by solving the characteristic polynomial, which can be calculated by hand. This approach is also acceptable but it is the most time consuming.

(d) With

$$\mathbf{x}_1 = \begin{pmatrix} \mathbf{e}_1 \\ a_1 \mathbf{e}_1 \end{pmatrix}$$

it is easy to check that

$$\mathbf{K}\mathbf{x}_1 = \begin{pmatrix} \mathbf{K}_3 \mathbf{e}_1 - a_1 \mathbf{e}_1 \\ -\mathbf{e}_1 + a_1 \mathbf{K}_3 \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 - a_1 \mathbf{e}_1 \\ -\mathbf{e}_1 + a_1 \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} (1 - a_1) \times \mathbf{e}_1 \\ \frac{a_1 - 1}{a_1} \times a_1 \mathbf{e}_1 \end{pmatrix}$$

which means that $\mathbf{x}_1$ is an eigenvector of $\mathbf{K}$ if

$$1 - a_1 = \frac{a_1 - 1}{a_1} = \hat{\lambda}_1.$$

The first equation can be solved with

$$a_1 = \pm 1.$$

The two corresponding eigenvectors and eigenvalues are

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_1 \end{pmatrix}, \lambda = 0, \quad \text{and} \quad \begin{pmatrix} \mathbf{e}_1 \\ -\mathbf{e}_1 \end{pmatrix}, \lambda = 2$$

Similarly, with

$$\mathbf{x}_2 = \begin{pmatrix} \mathbf{e}_2 \\ a_2 \mathbf{e}_2 \end{pmatrix}$$

it is easy to check that

$$\mathbf{K}\mathbf{x}_2 = \begin{pmatrix} \mathbf{K}_3 \mathbf{e}_2 - a_2 \mathbf{e}_2 \\ -\mathbf{e}_2 + a_2 \mathbf{K}_3 \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} 4\mathbf{e}_2 - a_2 \mathbf{e}_2 \\ -\mathbf{e}_2 + 4a_2 \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} (4 - a_2) \times \mathbf{e}_2 \\ \frac{4a_2 - 1}{a_2} \times a_2 \mathbf{e}_2 \end{pmatrix}$$

which means that $\mathbf{x}_2$ is an eigenvector of $\mathbf{K}$ if

$$4 - a_2 = \frac{4a_2 - 1}{a_2} = \hat{\lambda}_2.$$

The first of these equations can be solved with

$$a_2 = \pm 1.$$

The two corresponding eigenvectors and eigenvalues are

$$\begin{pmatrix} \mathbf{e}_2 \\ \mathbf{e}_2 \end{pmatrix}, \lambda = 3, \quad \text{and} \quad \begin{pmatrix} \mathbf{e}_2 \\ -\mathbf{e}_2 \end{pmatrix}, \lambda = 5$$

This provides 2 more eigenvectors/eigenvalues.

Exactly similar steps provide the final 2 eigenvectors/eigenvalues as

$$\begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_3 \end{pmatrix}, \lambda = 3, \quad \text{and} \quad \begin{pmatrix} \mathbf{e}_3 \\ -\mathbf{e}_3 \end{pmatrix}, \lambda = 5$$

10, D

4. (a) To show that $x = 0$ is grounded, consider

$$R = \frac{1 - z}{1 + z}.$$

On $x = 0$, where $\bar{z} = -z$, we have

$$\bar{R} = \frac{1 - \bar{z}}{1 + \bar{z}} = \frac{1 + z}{1 - z} = \frac{1}{R}$$

so that

$$|R|^2 = R\bar{R} = 1, \quad \text{on } x = 0.$$

Hence

$$\operatorname{Re}[h(z)] = \frac{V}{\log \rho} \log |R| = 0, \quad \text{on } x = 0.$$

Hence the boundary at $x = 0$ is “grounded” (at zero voltage).

3, A

- 4.(b) We need to show that the boundary of the electrode, which can be parametrized by

$$z = \frac{1 + \rho^2}{1 - \rho^2} + \frac{2\rho}{1 - \rho^2} e^{i\theta}, \quad 0 \leq \theta \leq 2\pi,$$

is at voltage V , i.e., $\phi = V$. From the given form of the voltage potential ϕ , and the properties of the complex logarithm ($\log z = \log |z| + i \arg[z]$) this is equivalent to showing that

$$\left| \frac{1 - z}{1 + z} \right| = \rho, \quad \text{or on squaring,} \quad \left| \frac{1 - z}{1 + z} \right|^2 = \rho^2$$

so that

$$\operatorname{Re}[h(z)] = \frac{V}{\log \rho} \log \left| \frac{1 - z}{1 + z} \right| = \frac{V}{\log \rho} \times \log \rho = V.$$

But it is easy to show that

$$1 - z = 1 - \left(\frac{1 + \rho^2}{1 - \rho^2} + \frac{2\rho}{1 - \rho^2} e^{i\theta} \right) = -\frac{2\rho}{1 - \rho^2} (\rho + e^{i\theta})$$

and

$$1 + z = 1 + \left(\frac{1 + \rho^2}{1 - \rho^2} + \frac{2\rho}{1 - \rho^2} e^{i\theta} \right) = \frac{2}{1 - \rho^2} (1 + \rho e^{i\theta}).$$

Consequently,

$$\left| \frac{1 - z}{1 + z} \right|^2 = \left(\frac{1 - z}{1 + z} \right) \times \left(\frac{1 - \bar{z}}{1 + \bar{z}} \right) = \rho^2 \frac{(\rho + e^{i\theta})(\rho + e^{-i\theta})}{(1 + \rho e^{i\theta})(1 + \rho e^{-i\theta})} = \rho^2 \frac{\rho^2 + 1 + 2\rho \cos \theta}{1 + \rho^2 + 2\rho \cos \theta} = \rho^2,$$

which is what we needed to show.

6, D

- 4.(c) The complex current density is given by

$$J^{(x)} - iJ^{(y)} = -\hat{c} \frac{dh}{dz} = -\frac{V}{\log \rho} \left(-\frac{1}{1 - z} - \frac{1}{1 + z} \right) = \frac{2V}{\log \rho (1 - z^2)},$$

where we have used the fact that $\hat{c} = 1$; the first equality is a relation derived in lectures.

Therefore on $x = 0$ where we can write $z = iy$ this is

$$J^{(x)} - iJ^{(y)} = \frac{2V}{\log \rho (1 + y^2)}.$$

Since this is purely real then $J^{(y)} = 0$ and hence the current is purely in the x direction, indeed if $V > 0$ then $J^{(x)}$ is negative because $0 \leq \rho < 1$ so $\log \rho < 0$.

3, A

4.(d) It is clear by inspection from the expression in part (c) that $J^{(x)}$ is greatest on the wall $x = 0$ at the point where $y = 0$, i.e., at the origin $(0, 0)$.

2, A

4.(e) The total current *leaving* the circuit through $x = 0$ is

$$\begin{aligned} \int_{-\infty}^{\infty} -J^{(x)} dy &= -\frac{2V}{\log \rho} \int_{-\infty}^{\infty} \frac{dy}{(1+y^2)} = -\frac{2V}{\log \rho} [\tan^{-1} y]_{-\infty}^{\infty} \\ &= -\frac{2V}{\log \rho} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = -\frac{2V\pi}{\log \rho}. \end{aligned}$$

3, A

4.(f) As $\rho \rightarrow 0$ with $V \rightarrow \infty$ with fixed

$$\frac{V}{\log \rho} = -\frac{m}{2\pi}$$

the radius of the electrode tends to zero while its centre tends to $(1, 0)$ or $z = 1$, and the complex potential tends to

$$h(z) = -\frac{m}{2\pi} \log \left(\frac{1-z}{1+z} \right) = -\frac{m}{2\pi} \log(z-1) + \frac{m}{2\pi} \log(z+1) + \text{constant}$$

which can be recognized as the complex potential for a point source of current of strength m at $z = 1$ (and a point source of current of opposite strength at $z = -1$ which is “inside” the grounded wall). Note that, in lectures, it was shown that if, near $z = z_0$,

$$h(z) = -\frac{m}{2\pi} \log(z - z_0) + \text{locally analytic function}$$

then we say there is a point current source of strength m at z_0 .

Not required: Note also that the total current through the grounded wall, from part (e), is

$$-\frac{2V\pi}{\log \rho} = -2\pi \times \frac{V}{\log \rho} = -2\pi \times \left(-\frac{m}{2\pi} \right) = m,$$

as expected.

3, B

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 80 of 80 marks

Total Mastery marks: 0 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode

Question# Comments for Students

The performance on this question was of a high standard, with generally good marks, and very good explanations.

In parts (a)-(c), many students missed the fact that, having found the required voltage, the effective conductance can be computed directly (simply by working out the net current out of the electrified node, which is easy if you know the voltages, by Ohm's law). Many students solved for the voltages, but then carried out a lot of (unnecessary) "circuit reductions" to find the effective conductance.

Parts (d) and (e) were found most difficult, with a few students not even attempting these. The idea of those final two parts is to show that you can use your answers to parts (a)-(c) to find the inverse of the reduced Laplacian matrix very easily. Many students missed this completely. Indeed, many realised part (e) was asking for the inverse of the reduced Laplacian and carried out a completely new calculation to find it. This is not what the question asked.

This three part question was generally well answered. There are different ways of finding the answer to part a) (algebra, counting loops) and the vast majority of students found the correct answer. Part b) required some careful algebra in addition to the specifics of circuits. A large number of students found the right answers. In the marking, I was forgiving with small algebraic mistakes and allowed for follow-on errors. There is an easy route to answering part c), but many students opted for a brute-force approach, which is lengthy and error-prone. I found myself often awarding partial marks for getting started correctly, even when things didn't work out at all towards the end. I even allowed for 2 numerical solutions, some of which were very creative.

This question was generally found to be straightforward. Nearly all were awarded full marks in Parts a, b. Part c too was generally done well by most candidates - using determinant or circulant matrix argument from lecture notes. Those who used direct determinant approach generally got the correct result. Those who used the circulant matrix approach, some candidates who had not practiced usage, made common mistakes, in interpreting how to use the formulae given in lecture notes -- marks were lost where real-valued eigenvalues were not computed, or some candidates simply left their answers in form $(1, w, w^*w)$ and $(1, w^{**2}, w^{**4})$ for the eigenvectors without evaluating the w 's for the repeated eigenvalue=4 --subsequently these candidates struggled with Part d too. Part d, was generally not answered as well as earlier parts. Common oversights in Part d, those who recognised that the results in Part c could be applied here, was to simply state eigenvectors in terms of the w 's, and in form $(e, -e)$ without explicit evaluation of the eigenvectors, however if a candidate stated that the w 's and e 's corresponded to those computed in Part c, then full marks were awarded. Some candidates did attempt to solve the 6th-order polynomial directly, with this route most then struggled, but there were some (surprisingly), who managed to compute the correct eigenvalues and eigenvectors - and thus were awarded full marks. Where it was obvious the candidate was aware of the correct technical approach for Part d (and generally) but some small slip up arose, the candidate in this case was not penalised too severely.

Parts 4a, c and d were generally done well. Part b was found to be most demanding, many candidates simply skipped this part. Major issue here was with candidates not knowing how to begin, or not being able to parameterise the circular electrode boundary correctly, followed on by the mere algebraic manipulations causing difficulties - with some candidates running into a couple of pages of reduction, while some managed same result with 3-4 lines of manipulation. Small slipups were not penalised significantly if it was evident that the overall approach was sound. Many candidates failed to recognise that all that was needed in Part b was to show that $|(1-z)/(1+z)| = \rho$. Part 4f, confused a number of candidates in terms of what was being asked - here : full marks were awarded if candidates spotted/stated that " recognised as complex potential for a point source at $z=1$, with $z=-1$ being analytic" and according to lecture notes.