

**BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024**

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

An Introduction to Applied Maths

Date: Monday, May 20, 2024

Time: 14:00 – 16:00 (BST)

Time Allowed: 2 hours

This paper has 4 Questions.

Please Answer Each Question in a Separate Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

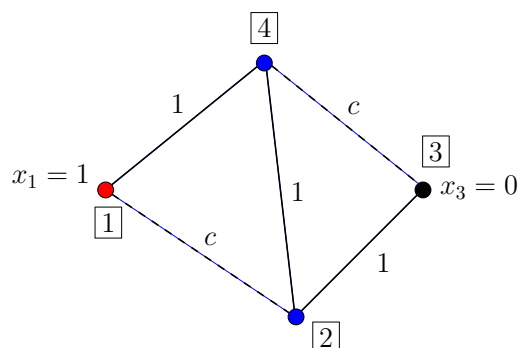
Allow margins for marking.

Credit will be given for all questions attempted.

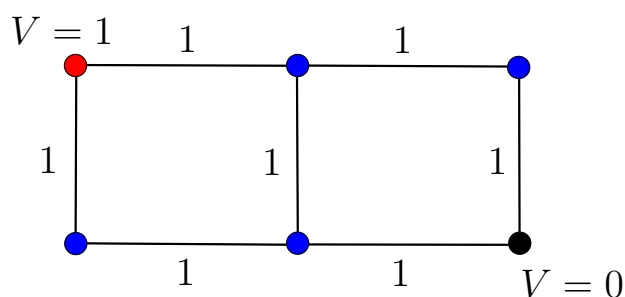
Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Consider the following electric circuit where the edge conductances are shown next to each edge: three of the conductors have unit conductance (solid lines) and two of the conductors have conductance c (dashed lines). Let x_i denote the voltage at node i . Node 1 is set to unit voltage ($x_1 = 1$), node 3 is grounded ($x_3 = 0$). Kirchhoff's current law holds at nodes 2 and 4 .



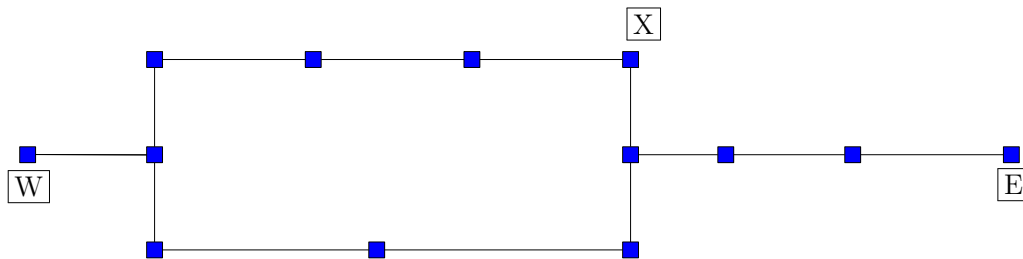
- Assigning columns/rows to nodes in the order $1\ 3\ 2\ 4$, find the conductance-weighted Laplacian matrix of this circuit. (2 marks)
- Find the effective conductance of this circuit in the limit $c \rightarrow 0$. (2 marks)
- Find the effective conductance of this circuit in the limit $c \rightarrow \infty$. (2 marks)
- Find the voltages at nodes 2 and 4 as a function of c . (5 marks)
- Find the effective conductance of the circuit for general values of c . Verify that your answers to (b) and (c) are retrieved in the appropriate limits. (4 marks)
- Use part (e) to find the effective conductance of the 6-node circuit below where the red node is at unit voltage $V = 1$, the black node is grounded $V = 0$, and Kirchhoff's current law holds at the 4 blue nodes:



(5 marks)

(Total: 20 marks)

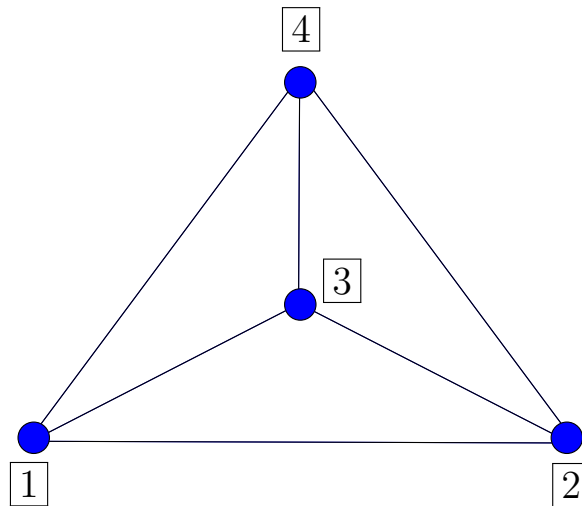
2. A tourist takes a random journey on the subway system shown in the figure: the blue boxes are stations and the lines between them are train lines. On arriving at each station, a random choice is made of which line to take next, with each choice having equal probability.



- (a) Consider the subway system as a graph with stations being nodes and the train lines being edges. Find the dimension of the left null space of any incidence matrix $\mathbf{A}$ of the graph and explain your answer (there is no need to write down an incidence matrix). (2 marks)
- (b) Find the dimension of the left null space of $\mathbf{A}^T \mathbf{A}$ and explain your answer. (2 marks)
- (c) Redraw the graph but with a **single** edge removed so that the dimension of the left null space of $\mathbf{A}^T \mathbf{A}$ increases by 1. (2 marks)
- (d) Returning now to the full subway system shown above (with no edges removed), find the probability that the tourist starting a journey at station $\boxed{W}$ arrives at station $\boxed{E}$ before returning to station $\boxed{W}$. (7 marks)
- (e) The tourist starts a random journey at station $\boxed{X}$. Find the probability the tourist reaches station $\boxed{E}$ before station $\boxed{W}$. (7 marks)

(Total: 20 marks)

3. Consider a complete graph with 4 nodes.



- (a) Write down the graph Laplacian matrix $\mathbf{K}$ using the given node numbering to label corresponding columns/rows of $\mathbf{K}$. (2 marks)
- (b) Suppose that the graph is an electric circuit with all edges having unit conductance. Find the effective conductance when node $\boxed{1}$ is set to unit voltage and node $\boxed{2}$ is grounded. (4 marks)
- (c) Find the set of real-valued orthonormal eigenvectors $\{\mathbf{x}_j | j = 1, 2, 3, 4\}$ and corresponding eigenvalues $\{\lambda_j | j = 1, 2, 3, 4\}$ of $\mathbf{K}$. (6 marks)
- (d) Suppose now that a unit **current** is fed into node $\boxed{1}$ and node $\boxed{2}$ is grounded (set to zero voltage). Let $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ be the associated 4-dimensional vector of node voltages in this case where x_i is the voltage at node $\boxed{i}$. Find the coefficients $\{a_j | j = 1, 2, 3, 4\}$ in the formula

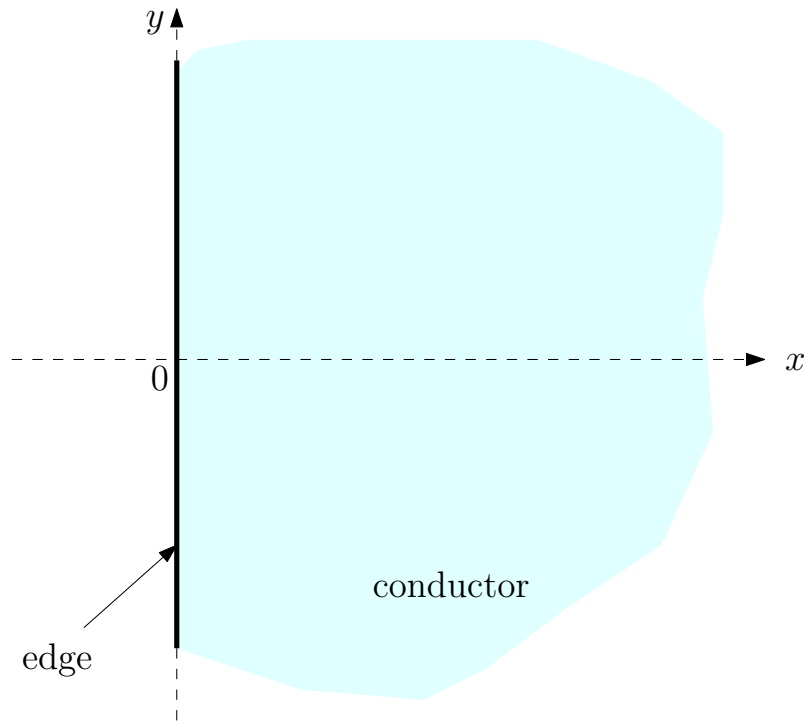
$$\mathbf{x} = \sum_{j=1}^4 a_j \mathbf{x}_j,$$

where $\{\mathbf{x}_j | j = 1, 2, 3, 4\}$ are the orthonormal eigenvectors found in part (c). (6 marks)

- (e) Use your answer to part (d) to check your answer to part (b). (2 marks)

(Total: 20 marks)

4. Consider a two-dimensional conductor of unit conductivity occupying the whole of the right half $x \geq 0$ of the (x, y) -plane with an edge along $x = 0$ as indicated in the figure:



The voltage distribution $\phi(x, y)$ at any point (x, y) in the conductor is given by

$$\phi(x, y) = \operatorname{Re}[h(z)], \quad h(z) = -\frac{m}{2\pi} \log \left(\frac{(1-z)(2+z)}{(1+z)(2-z)} \right), \quad z = x + iy,$$

where $i = \sqrt{-1}$, m is a positive real constant, and $\operatorname{Re}[\cdot]$ denotes the real part of the complex-valued quantity in square brackets.

- Show that $\phi = 0$ along the edge $x = 0$. (2 marks)
- Show that this voltage distribution is associated with two point current sources in the conductor and find their locations and their strengths. (2 marks)
- Find an expression for the current density $J^{(x)} - iJ^{(y)}$. (3 marks)
- Show that $J^{(y)} = 0$ along the edge $x = 0$. (1 mark)
- Find a formula, as a function of y , for the current density $J^{(x)}$ along the edge $x = 0$. (2 marks)
- Calculate the total current leaving the conductor through the edge $x = 0$. (4 marks)
- Sketch a graph of the quantity $(\pi/m)J^{(x)}$ on the edge $x = 0$ as a function of y . (6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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M40007

Introduction to Applied Mathematics (Solutions)

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1. (a) Assigning columns/rows to nodes in the order $\boxed{1}\boxed{3}\boxed{2}\boxed{4}$, the conductance-weighted Laplacian can be calculated using the usual rules to be

sim. seen ↓

$$\mathbf{K} = \begin{pmatrix} c+1 & 0 & -c & -1 \\ 0 & c+1 & -1 & -c \\ -c & -1 & c+2 & -1 \\ -1 & -c & -1 & c+2 \end{pmatrix}$$

2, A

- (b) When $c = 0$ the circuit reduces to 3 unit-conductance conductors in series with effective conductance C_{eff} satisfying

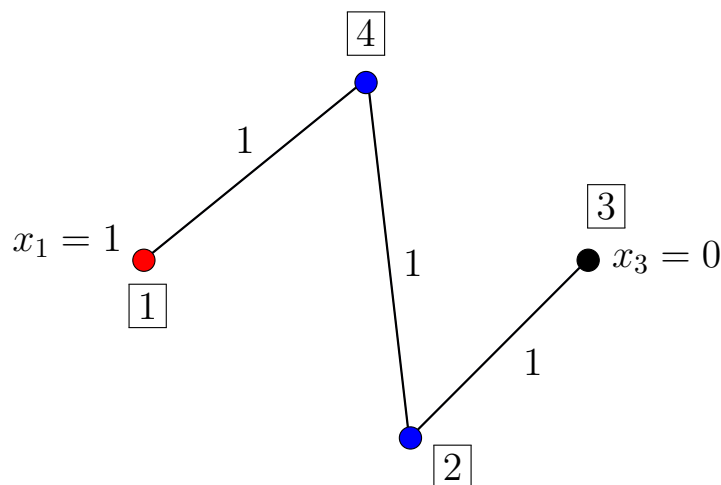
sim. seen ↓

$$\frac{1}{C_{\text{eff}}} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1} = 3$$

so that

$$C_{\text{eff}} = \frac{1}{3}.$$

Here we have used the fact that *resistances* in series add.



2, A

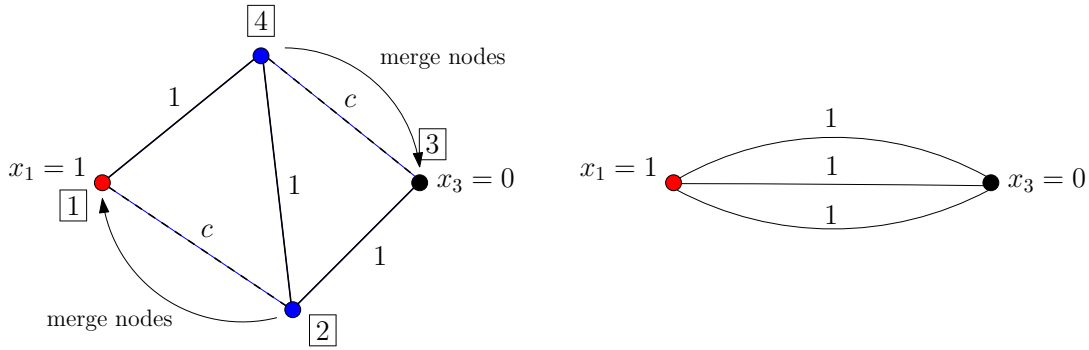
- (c) When $c \rightarrow \infty$, if the currents are to remain bounded then the voltage drop at the two nodes connected by these edges must tend to zero (the nodes must be at the same voltage) meaning that these adjacent nodes can be combined as shown below to produce circuit with 3 unit-conductance conductors in parallel with effective conductance

sim. seen ↓

$$C_{\text{eff}} = 3.$$

Here we have used the fact that *conductances* in parallel add.

2, B



- (d) To find the voltage X at node $\boxed{4}$ and the voltage Y at node $\boxed{2}$ the easiest approach is to use KCL at each of these two nodes (this is equivalent to harmonicity at these nodes). At node $\boxed{4}$, the current in from node $\boxed{1}$ is $1 - X$, and the currents out to nodes $\boxed{2}$ and $\boxed{3}$ are respectively $X - Y$ and cX . Balancing these gives

sim. seen $\Downarrow$

$$1 - X = X - Y + cX \quad (1)$$

implying that

$$Y = (2 + c)X - 1. \quad (2)$$

Similarly, at node $\boxed{2}$, the current in from node $\boxed{1}$ is $c(1 - Y)$, and the currents out to nodes $\boxed{4}$ and $\boxed{3}$ are respectively $Y - X$ and Y . Balancing these gives

$$c(1 - Y) = Y - X + Y \quad (3)$$

implying that

$$X = (2 + c)Y - c. \quad (4)$$

On substitution for Y from (2),

$$X = (2 + c)((2 + c)X - 1) - c. \quad (5)$$

On rearrangement,

$$X = \frac{2(1 + c)}{(2 + c)^2 - 1}. \quad (6)$$

It can be checked, on back substitution into (2), that

$$Y = 1 - X \quad (7)$$

which can also be deduced by a symmetry argument.

Extra note: Another approach is to use the conductance-weighted Laplacian, $\mathbf{K}$ say, from part (a) and to solve

$$\mathbf{K}\mathbf{x} = \mathbf{f}, \quad (8)$$

where

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ Y \\ X \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} C_{\text{eff}} \\ -C_{\text{eff}} \\ 0 \\ 0 \end{pmatrix}.$$

- (e) The effective conductance is the net current into node 3 (or equivalently the net current out of node 1) which is

sim. seen ↓

$$C_{\text{eff}} = cX + Y. \quad (9)$$

This equals

$$C_{\text{eff}} = cX + (1 - X) = (c - 1)X + 1 = \frac{2(c^2 - 1)}{(2 + c)^2 - 1} + 1 = \frac{3c^2 + 4c + 1}{c^2 + 4c + 3} \quad (10)$$

Now we can check the limits $c \rightarrow 0, \infty$. As $c \rightarrow 0$, it is clear that

$$C_{\text{eff}} \rightarrow \frac{1}{3} \quad (11)$$

in agreement with part (b).

As $c \rightarrow \infty$, it is clear that

$$C_{\text{eff}} \rightarrow 3 \quad (12)$$

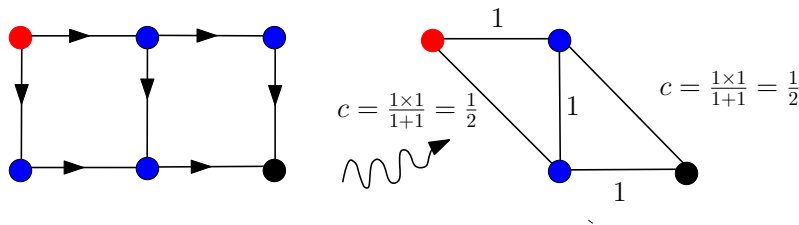
in agreement with part (c).

4, D

- (f) By using the rule for two unit-conductance conductors in series, twice, the given circuit reduces to an equivalent circuit as above with the choice $c = 1/2$. Hence,

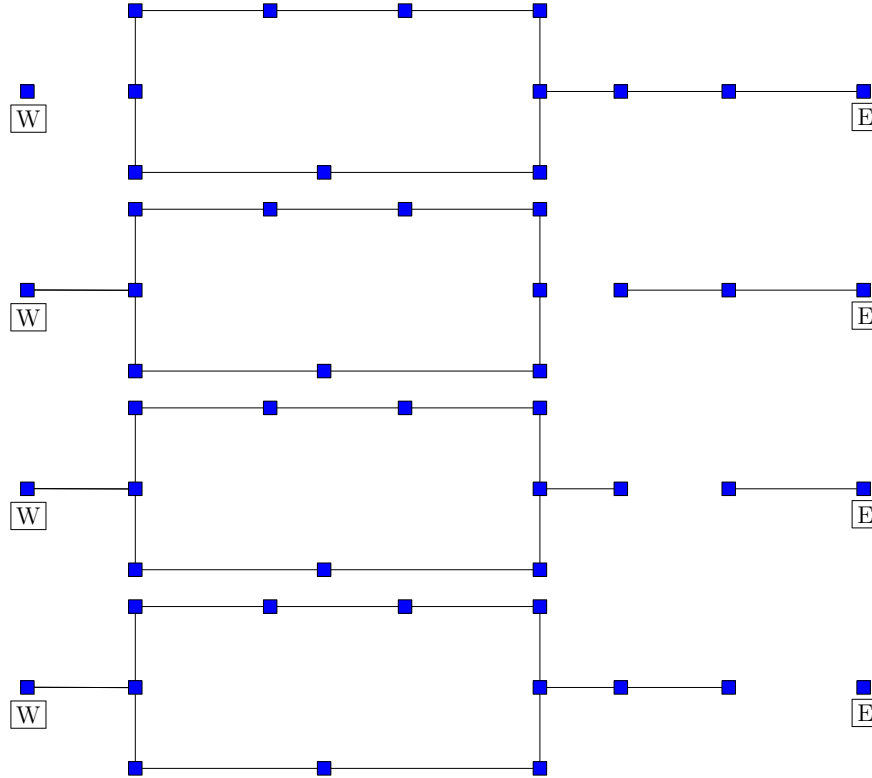
sim. seen ↓

$$C_{\text{eff}} = \frac{3/4 + 2 + 1}{1/4 + 2 + 3} = \frac{5}{7} \quad (13)$$



5, B

2. (a) The dimension of the left null space of an incidence matrix is the number of non-trivial closed loops in the graph, which is clearly 1.
- (b) The matrix $\mathbf{A}^T \mathbf{A}$ is symmetric, so its left null space is the same as its right null space. The graph is connected, so $\mathbf{A}$ has a single null vector comprising a vector with all equal elements.
- (c) Any one of the edges that are **not** in the “loop” portion of the graph can be removed and the graph would become disconnected, thus increasing the dimension of the right null space of $\mathbf{A}^T \mathbf{A}$ by 1. The 4 possibilities are shown below (any one of these suffices as an answer):



- (d) This question asks for the escape probability p_{esc} to station $\boxed{\text{E}}$ from station $\boxed{\text{W}}$. By the analogy with electric circuits, and since there is only one edge connected to station $\boxed{\text{W}}$, this is equal to the effective conductance C_{eff} in an equivalent electric circuit with all edges conductances taken equal (without loss of generality, to unity) and where station $\boxed{\text{W}}$ is set to unit voltage and station $\boxed{\text{E}}$ is grounded. This follows from the result given in lectures

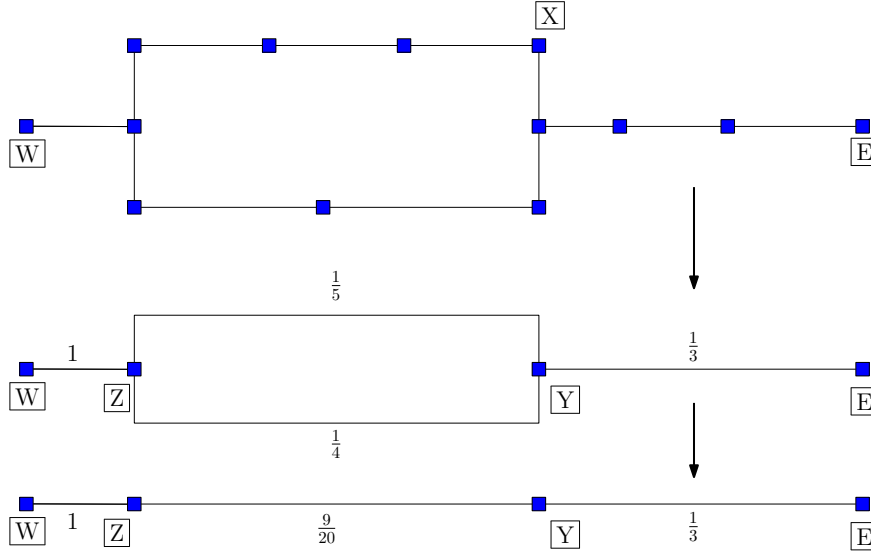
$$p_{\text{esc}} = \frac{C_{\text{eff}}}{c_+}, \quad (14)$$

where c_+ is the total number of (unit conductance) edges attached to station $\boxed{\text{W}}$ (here $c_+ = 1$). The value of C_{eff} can be found by the method of equivalent circuits by repeated use of the formulas for two conductors in parallel $C_{\text{eff}} = c_a + c_b$ and two in series $C_{\text{eff}} = \frac{c_a c_b}{c_a + c_b}$. From the final equivalent circuit in the sequence shown below

$$\frac{1}{C_{\text{eff}}} = 1 + \frac{20}{9} + 3 = \frac{9 + 20 + 27}{9} = \frac{56}{9} \quad (15)$$

or

$$C_{\text{eff}} = \boxed{\frac{9}{56}} \quad (16)$$



7, B

sim. seen ↓

- (e) By the same electric circuit analogy, this question asks for the voltage at node $\boxed{X}$ if node $\boxed{E}$ is set to unit voltage and node $\boxed{W}$ is grounded. The effective conductance is the same as calculated in part (d) (it does not depend on which of the two nodes is set to unit voltage) therefore the current out of node $\boxed{E}$ is $9/56$ (by the definition of the effective conductance). Therefore, by Ohm's law, the voltage at node $\boxed{Y}$, denoted by Y , satisfies

$$\frac{1}{3}(1 - Y) = \frac{9}{56}, \quad \text{or} \quad Y = \frac{29}{56}. \quad (17)$$

Similarly, the voltage at node $\boxed{Z}$ is

$$Z - 0 = Z = \frac{9}{56} \quad (18)$$

which follows by Ohm's law since the same current of $9/56$ must be flowing in the (unit conductance) edge connecting node $\boxed{Z}$ to node $\boxed{W}$. Therefore since the northern loop has effective conductance $1/5$ the current in it is

$$\frac{1}{5}(Y - Z) = \frac{1}{5} \times \left(\frac{29}{56} - \frac{9}{56} \right) = \frac{4}{56}, \quad (19)$$

where we have again used Ohm's law. By Ohm's law again the voltage at $\boxed{X}$ therefore satisfies

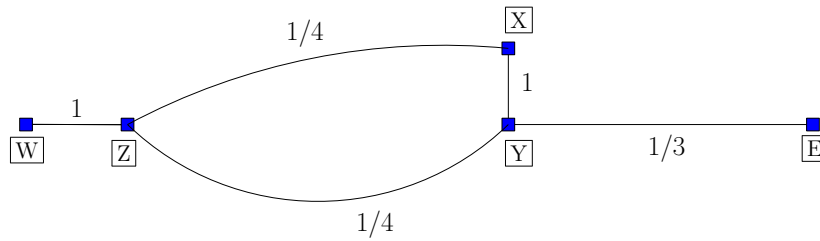
$$Y - X = \frac{4}{56}, \quad (20)$$

where Ohm's law has been used again (the conductance between $\boxed{X}$ and $\boxed{Y}$ is unity) therefore

$$X = Y - \frac{4}{56} = \frac{29}{56} - \frac{4}{56} = \boxed{\frac{25}{56}}. \quad (21)$$

This is the required probability.

7, C



Extra note: The above is the easiest way to answer part (e) but another approach is to think of the circuit as equivalent to the one shown in the figure.

The KCL condition at node $\boxed{X}$, using X to denote the voltage there, is

$$\frac{5}{4}X = Y + \frac{Z}{4}, \quad \text{or} \quad 5X = 4Y + Z \quad (22)$$

where we have written down a “weighted harmonicity” condition. Doing the same thing at $\boxed{Y}$ gives

$$\left(1 + \frac{1}{3} + \frac{1}{4}\right)Y = \frac{1}{3} + X + \frac{Z}{4} \quad (23)$$

or

$$19Y = 4 + 12X + 3Z \quad (24)$$

and at $\boxed{Z}$ we get

$$\left(1 + \frac{1}{2}\right)Z = \frac{X}{4} + \frac{Y}{4}, \quad \text{or} \quad 6Z = X + Y. \quad (25)$$

To find X the 3 equations (22), (24) and (25) must be solved. From (22) and (25)

$$\begin{aligned} Z &= 5X - 4Y \\ Z &= \frac{X}{6} + \frac{Y}{6}. \end{aligned} \quad (26)$$

Subtraction gives

$$Y = \frac{29X}{25}. \quad (27)$$

Use of this in (25) then leads to

$$Z = \frac{1}{6} \left(X + \frac{29}{25}X \right) = \frac{9X}{25}. \quad (28)$$

Finally, use of (27) and (28) in (24) gives

$$19 \times \frac{29X}{25} = 4 + 12X + \frac{27X}{25} \quad (29)$$

or

$$X = \frac{100}{19 \times 29 - 12 \times 25 - 27} = \frac{100}{551 - 300 - 27} = \frac{100}{224} = \boxed{\frac{25}{56}}. \quad (30)$$

3. (a) By the usual construction,

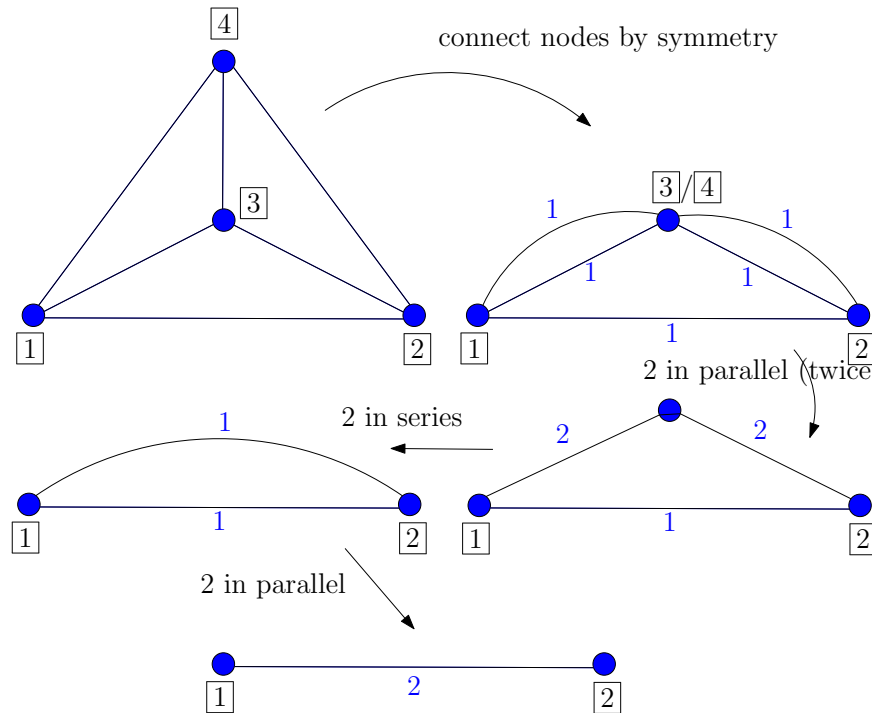
sim. seen ↓

$$\mathbf{K} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

2, A

- (b) The effective conductance can be obtained by the method of equivalent circuits once symmetry is used to deduce that nodes 3 and 4 can be combined (while preserving all connected edges) because they must be at the same voltage (there is no reason more current would go through one than the other):

sim. seen ↓



Here we have repeatedly used the rule for two conductors in parallel ($C_{\text{eff}} = c_a + c_b$) and in series ($C_{\text{eff}} = c_a c_b / (c_a + c_b)$). The required effective conductance is

$$C_{\text{eff}} = 2.$$

4, A

- (c) By inspection, the matrix $\mathbf{K}$ is circulant so complex-valued eigenvectors are

sim. seen ↓

$$\mathbf{X}_j = \begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ \omega^{3j} \end{pmatrix}, \quad \omega = e^{2\pi i/4} = i, \quad j = 0, 1, 2, 3.$$

This result is given on the handout. These are, for $j = 0, 1, 2, 3$,

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ +i \\ -1 \\ -i \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -i \\ -1 \\ +i \end{pmatrix},$$

with corresponding eigenvalues, obtained by multiplying these vectors by $\mathbf{K}$ and inspecting the first element of the result, i.e.

$$3 - \omega^j - \omega^{2j} - \omega^{3j} \quad (31)$$

which give the values

$$0, \quad 4, \quad 4, \quad 4.$$

Not all these eigenvectors are real, however. Taking real and imaginary parts of the complex-valued ones gives the required real-valued set

$$\mathbf{x}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},$$

where we have also made these vectors *orthonormal* by rescaling. The corresponding eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = 4, \quad \lambda_3 = 4, \quad \lambda_4 = 4.$$

(d) The governing equation is

$$\mathbf{K}\mathbf{x} = \mathbf{f} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

since the unit current entering node 1 must exit at node 2. Notice that

$$\mathbf{K}\mathbf{x} = \mathbf{K} \sum_{j=1}^4 a_j \mathbf{x}_j = \sum_{j=1}^4 a_j \lambda_j \mathbf{x}_j = \mathbf{f} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

We can multiply this by $\mathbf{x}_2^T$, $\mathbf{x}_3^T$ and $\mathbf{x}_4^T$ to get, respectively,

$$\sum_{j=1}^4 a_j \lambda_j \mathbf{x}_2^T \mathbf{x}_j = \mathbf{x}_2^T \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = 1, \quad a_2 \lambda_2 = 1, \quad a_2 = \frac{1}{4},$$

$$\sum_{j=1}^4 a_j \lambda_j \mathbf{x}_3^T \mathbf{x}_j = \mathbf{x}_3^T \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}, \quad a_3 \lambda_3 = \frac{1}{\sqrt{2}}, \quad a_3 = \frac{1}{4\sqrt{2}},$$

and

$$\sum_{j=1}^4 a_j \lambda_j \mathbf{x}_4^T \mathbf{x}_j = \mathbf{x}_4^T \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{\sqrt{2}}, \quad a_4 \lambda_4 = -\frac{1}{\sqrt{2}}, \quad a_4 = -\frac{1}{4\sqrt{2}}.$$

6, D

sim. seen ↓

In the above statements, the orthonormality of the eigenvectors has been used. Therefore,

$$\mathbf{x} = a_1 \times \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{4} \times \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{4\sqrt{2}} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{4\sqrt{2}} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

But since node 2 is grounded, the second element must equal zero, i.e.,

$$0 = \frac{a_1}{2} - \frac{1}{8} - \frac{1}{8}, \quad a_1 = \frac{1}{2}.$$

The final result is

$$\mathbf{x} = \boxed{\frac{1}{2}} \times \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \boxed{\frac{1}{4}} \times \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \boxed{\frac{1}{4\sqrt{2}}} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \boxed{-\frac{1}{4\sqrt{2}}} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

6, B

(e) From part (d), the voltage at node 1 is

sim. seen ↓

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}.$$

Therefore, multiplying all voltages and currents by 2 will produce the circuit with node 1 set to **unit** voltage and node 2 grounded. In that case the current out of node 1 will be 2. This is the effective conductance found in part (b), providing a check on that solution.

2, A

4. (a) Let

sim. seen ↓

$$R(z) = \frac{(1-z)(2+z)}{(1+z)(2-z)}$$

and suppose that $z = iy$ corresponding to a point on the edge $x = 0$. Then, on this edge,

$$R(z) = \frac{(1-iy)(2+iy)}{(1+iy)(2-iy)} = \frac{(1-iy)}{(1+iy)} \times \frac{(2+iy)}{(2-iy)},$$

where both terms in this final expression are a ratio of complex conjugate quantities so that each has unit modulus, i.e.,

$$\left| \frac{(1-iy)}{(1+iy)} \right| = 1, \quad \left| \frac{(2+iy)}{(2-iy)} \right| = 1.$$

As a result, $|R| = 1$ and

$$\phi = \text{Re} \left[-\frac{m}{2\pi} \log R(z) \right] = -\frac{m}{2\pi} \log |R(z)| = 0, \quad \text{on } x = 0. \quad (32)$$

2, A

(b) Notice that, using well known properties of the logarithm, $h(z)$ can be written as

sim. seen ↓

$$h(z) = -\frac{m}{2\pi} \log(z-1) + \frac{m}{2\pi} \log(z-2) + \hat{h}(z) \quad (33)$$

where

$$\hat{h}(z) = -\frac{m}{2\pi} \log \left(\frac{2+z}{1+z} \right) \quad (34)$$

has no singularities in the conductor $x \geq 0$. The first term on the right hand side of (33) corresponds to a point source of current of strength m at $z = 1$, the second term corresponds to a point source of current of strength $-m$ at $z = 2$.

2, A

(c) Using the formula from lectures

sim. seen ↓

$$J^{(x)} - iJ^{(y)} = -\hat{c} \frac{dh}{dz} = \frac{m}{2\pi} \left[\frac{1}{z-1} + \frac{1}{z+2} - \frac{1}{z+1} - \frac{1}{z-2} \right], \quad (35)$$

where we have set the conductivity $\hat{c} = 1$. Thus

$$J^{(x)} - iJ^{(y)} = \frac{m}{2\pi} \left[\frac{2}{z^2-1} - \frac{4}{z^2-4} \right]. \quad (36)$$

3, A

(d) On the edge $x = 0$, $z = iy$, so the previous expression becomes

sim. seen ↓

$$J^{(x)} - iJ^{(y)} = \frac{m}{2\pi} \left[-\frac{2}{1+y^2} + \frac{4}{4+y^2} \right] \quad (37)$$

which is purely real, so $J^{(y)} = 0$.

1, A

(e) It can also be read off from the previous formula that, on $x = 0$,

sim. seen ↓

$$J^{(x)} = \frac{m}{\pi} \left[-\frac{1}{1+y^2} + \frac{2}{4+y^2} \right]. \quad (38)$$

2, A

- (f) The total current out of the conductor through the edge at $x = 0$ is

sim. seen ↓

$$-\int_{-\infty}^{\infty} J^{(x)} dy = -\frac{m}{\pi} \int_{-\infty}^{\infty} \left[-\frac{1}{1+y^2} + \frac{2}{4+y^2} \right] dy \quad (39)$$

Letting $y = 2q$ in the second integral:

$$-\int_{-\infty}^{\infty} J^{(x)} dy = -\frac{m}{\pi} \int_{-\infty}^{\infty} \left[-\frac{dy}{1+y^2} + \frac{2(2dq)}{4+4q^2} \right] = 0. \quad (40)$$

4, A

- (g) First note that $(\pi/m)J^{(x)}$ is an even function of y . Also, it decays as $|y| \rightarrow \pm\infty$. Note that

sim. seen ↓

$$\begin{aligned} \frac{\pi}{m} \frac{dJ^{(x)}}{dy} &= \left[\frac{2y}{(1+y^2)^2} - \frac{4y}{(4+y^2)^2} \right] = 2y \left[\frac{(4+y^2)^2 - 2(1+y^2)^2}{(1+y^2)^2(4+y^2)^2} \right] \\ &= -2y \left[\frac{y^4 - 4y^2 - 14}{(1+y^2)^2(4+y^2)^2} \right]. \end{aligned} \quad (41)$$

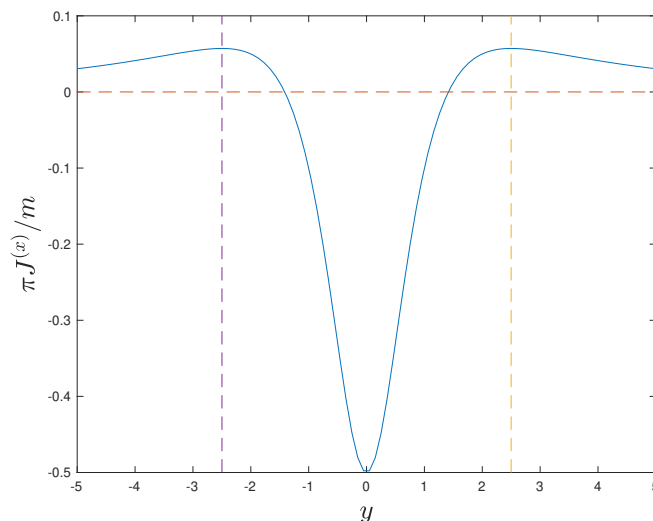
Since this is negative as $y \rightarrow +\infty$ then the curve approaches zero from above. By the continuity of the function, and the fact that it is even, there must be local maxima at

$$y^2 = \frac{4 + \sqrt{16 + 56}}{2} = \frac{4 + \sqrt{72}}{2}, \quad y = \pm \sqrt{\frac{4 + \sqrt{72}}{2}} \quad (42)$$

and a local minimum at $y = 0$ where $(\pi/m)J^{(x)} = -1/2$. The local maxima must be positive since the total current out of this edge, which is the total (signed) area between this curve and the axis, is zero, by part (e). In fact, these maxima are

$$\left[-\frac{1}{1+y^2} + \frac{2}{4+y^2} \right]_{y^2 = \frac{4+\sqrt{72}}{2}} = \frac{4}{12 + \sqrt{72}} - \frac{2}{6 + \sqrt{72}} \quad (43)$$

although, in a sketch, it is not necessary to calculate this. Correctly sketching qualitative features of the following graph will be awarded full marks:



6, D

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 80 of 80 marks

Total Mastery marks: 0 of 20 marks

Question Marker's comment

- 1 This question was done very well, with lots of full mark scores. The most difficult part was part (c), where the infinite conductances caught out many students. The idea was that part (b) and (c) could be done by recognising that the circuits can be reduced, in these special cases, to standard "series" and "parallel" circuits where the effective conductance can be computed directly. Many students spotted this, but many did not but, in any case, full marks were awarded to anybody who found the correct answers, no matter which method was used. Overall, I was pleased to see a strong command of the ideas by the student body.

- 3 Answered well overall but many mistakes were made by not fully reading the question. Part (a) was answered correctly by all who attempted it. Part (b) was sometimes a little brief, without any intermediate steps or explanation, but on the whole most students answered very well. Many people did not read the question carefully in part (c), in which it was specified that a real, orthonormal eigenbasis was required. A large number of students did not find real vectors or normalise their vectors. This lost marks and made subsequent parts far more difficult. Some used the wrong formula from the booklet. Part (d) was answered well, with many who answered (c) correctly also answering (d) correctly. Part (d) specified an orthonormal eigenbasis should be used, if not one mark was deducted, but otherwise full marks could be achieved regardless of the answer to part (c). Some people explained the method without implementing it (gaining partial marks) and mistakes were made in finding the coefficient for the vector with zero eigenvalue. Some people did not notice the question specified unit current not unit voltage and proceeded to solve the same problem as part (b) or used part (b) in their answer. This meant parts (b) and (d) automatically agreed, trivialising part (e). Part (e) was skipped by many but most who answered got at least one mark. There was some confusion about the difference between C_{eff} and f_1 , leading to some messy explanations.